MODULE 5

MATHEMATICS 30

Quadratic Relations

Distance Learning

Alberta EDUCATION

Mathematics 30

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QUADRATIC RELATIONS

Distance Learning

Alberta EDUCATION

This document is intended for	
Students	✓
Teachers (Mathematics 30)	✓
Administrators	
Parents	
General Public	
Other	

Mathematics 30
Student Module Booklet
Module 5
Quadratic Relations
Alberta Distance Learning Centre
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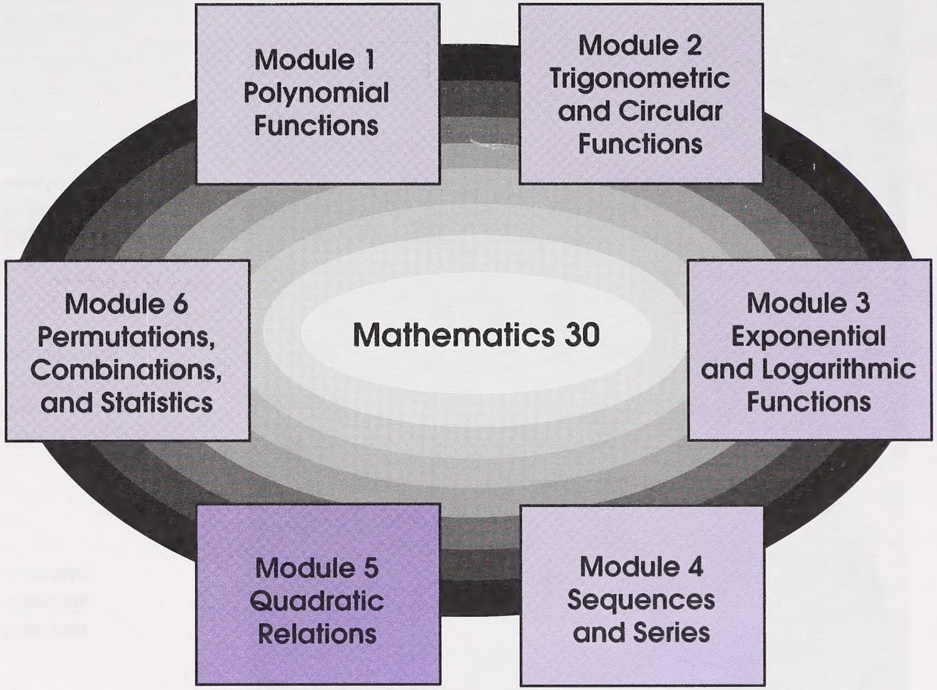
Welcome

WESTFILE INC.

Welcome to Module 5. We hope you'll enjoy your study of Quadratic Relations.

Mathematics 30 contains six modules. Work through the modules in the order given, since several concepts build on each other as you progress in the course.



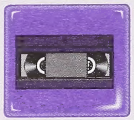
The document you are presently reading is called a Student Module Booklet. You may find visual cues or icons throughout it. Read the following explanations to discover what each icon prompts you to do.

- Use your graphing calculator.

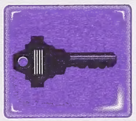
- Use your scientific calculator.

- Use computer software.

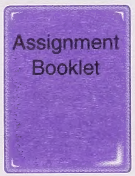
- Use the suggested answers in the Appendix to correct the activities.



- View a videocassette.

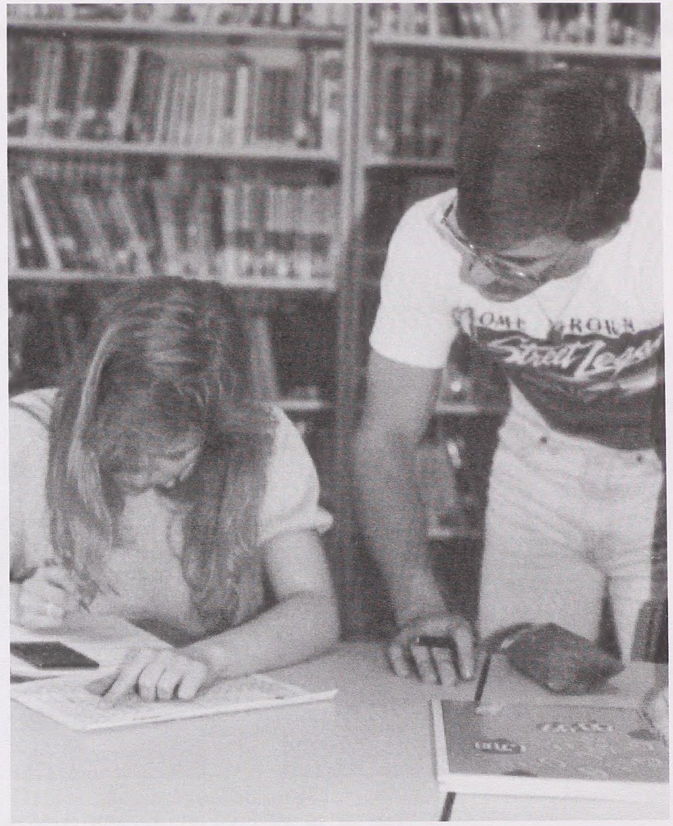


- Pay close attention to important words or ideas.



- Answer the questions in the Assignment Booklet.

Note: Whenever the scientific calculator icon appears, you may use a graphing calculator instead.



There are no response spaces provided in this Student Module Booklet. This means that you will need to use your own paper for your responses. You should keep your response pages in a binder so that you can refer to them when you are reviewing or studying.

CONTENTS

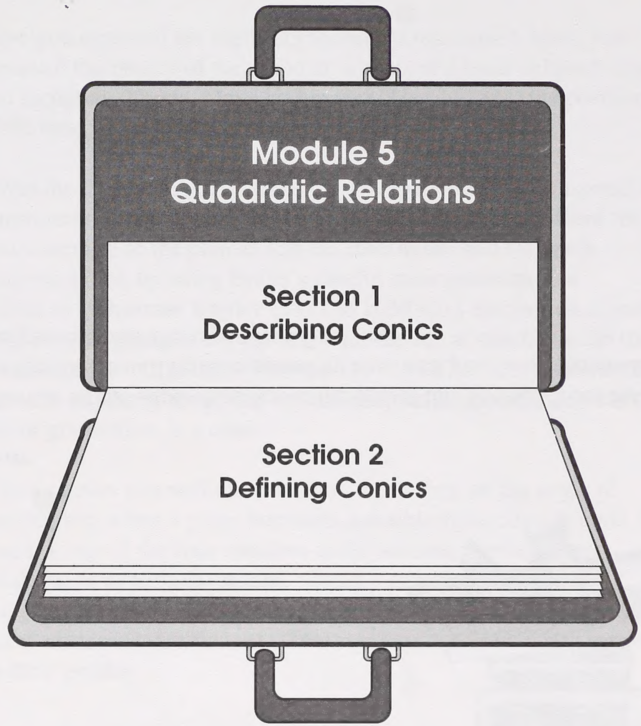
Module Overview

The word *geometry* comes from the Greek word *geometria*, meaning **to measure the earth**. Implied in this phrase is the notion that the elements of the physical universe can be categorized by form and size. For instance, the growth rings of a tree form concentric circles, grain poured onto a pile forms an inverted cone, water cascading over a waterfall traces out a parabola, and the outline of a thumbprint is an ellipse.

The study of quadratic relations links the geometric forms (circle, ellipse, parabola, and hyperbola) that can be obtained by slicing a cone with the tools of algebra. As a result, you will be able to explore real-world applications of these curves.

Opening up Module 5, you will discover that the topic of quadratic relations has been divided into two sections.

Section 1 deals with both a physical description of conics and how their attributes change with changes in the coefficients of the quadratic equations that represent those conics. Section 2 investigates two ways of defining the curves: the locus definitions and eccentricity. In both sections, as you leaf through the activities, you will find applications to the sciences and technology—from the orbits of the planets to the design of car headlamps.



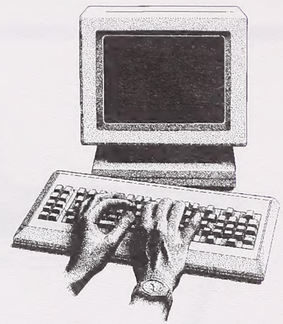
Evaluation

Your mark for this module will be determined by how well you complete the assignments at the end of each section and at the end of the module. In this module you must complete two section assignments and one final module assignment. The mark distribution is as follows:

Section 1 Assignment	**38 marks**
Section 2 Assignment	**36 marks**
Final Module Assignment	**26 marks**
TOTAL	**100 marks**

When doing the assignments, work slowly and carefully. You must do each assignment independently; but if you are having difficulties, you may review the appropriate section in this module booklet.

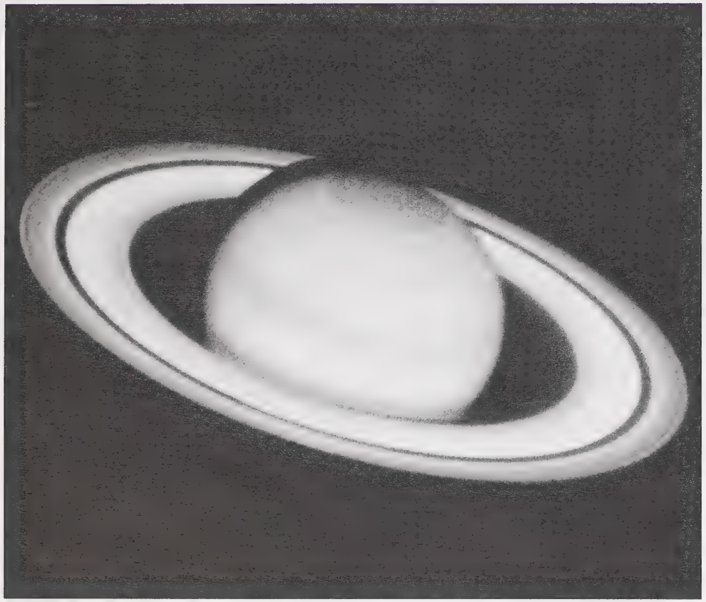
If you are working on a CML terminal, you will have a module test as well as a module assignment.



Note

There is a final supervised test at the end of this course. Your mark for the course will be determined by how well you do on the module assignments and the supervised final test.

Section 1: Describing Conics



NASA

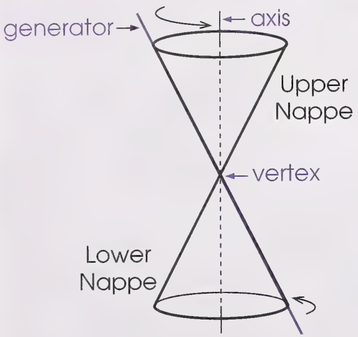
Have you explored the night sky through a telescope? Have you observed the phases of the moon or witnessed a lunar eclipse? Can you recognize Venus, Mars, or Jupiter? Do you know the positions of the other planets?

It was the change in positions of the celestial bodies that inspired an astronomer named **Tycho Brahe** (1546 to 1601 A.D.) to record the exact location of the planets and the stars in the mid sixteenth century. Later, by using Brahe's careful measurements as a guideline, **Johannes Kepler** (1571 to 1630 A.D.) discovered ellipses to be the true paths of the planets, with the sun at one focus. In the seventeenth century, **Isaac Newton** (1642 to 1727 A.D.) proved that the orbit of a body revolving around another, in accordance with the law of gravitation, is a conic.

In this section you will discover that, depending on the angle of intersection, when a plane intersects a double-right circular cone, the shape of one of the four standard conic sections (circle, ellipse, parabola, or hyperbola) results. Using a graphing calculator or computer program, you will also investigate and describe the effects of the numerical coefficients of the equations of these conic sections on their graphs.

Activity 1: Physical Properties of Conic Sections

About 200 B.C., a Greek mathematician, **Apollonius**, discovered the shapes that occur when a **double-napped** cone is intersected by a plane. He pictured the cones as two opposite **nappes** that are connected at their respective vertices and extend away infinitely. The two portions are referred to as the upper nappe and the lower nappe. This is illustrated in the following diagram of a double-napped cone.

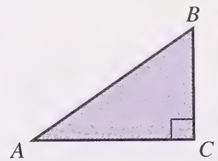


You can also think of the nappes as being formed by rotating an oblique line about a vertical axis. When you do this, the oblique line is referred to as the **generator**.

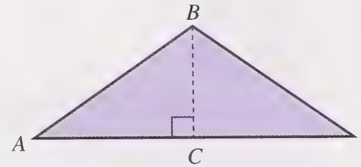
The shapes described by the circle, parabola, ellipse, and hyperbola are often called conics.

A **conic** is obtained when a plane intersects a double-napped cone. The intersection may occur horizontally, vertically, or obliquely (at a slant).

Consider the right triangle ABC which follows. You will notice that a cone can be generated if the triangle is revolved about the side BC.

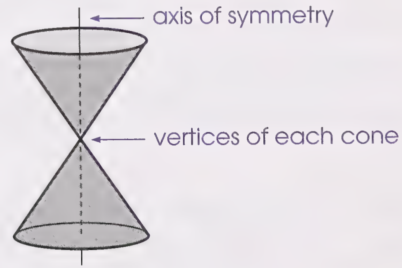


Right ΔABC

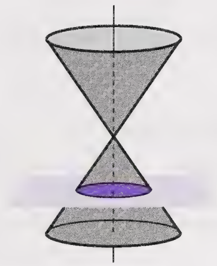


A cross section of a single circular cone is produced when ΔABC is rotated about side BC.

The double-napped cone is obtained by positioning two right-circular cones so that their vertices coincide. Recall the diagram of a double-napped cone.

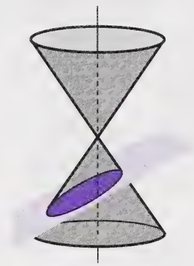


When a double-napped cone is intersected by a plane at right angles to its axis, the cross section is a **circle**.



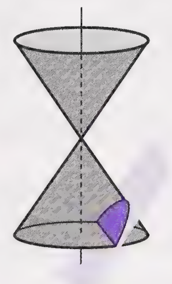
Circle

If the plane intersects one nappe of a double-napped cone at neither a right angle to the axis nor parallel to a generator, then the cross section is an **ellipse**.



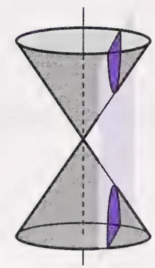
Ellipse

If the plane intersects the double-napped cone parallel to a generator, the cross section is a **parabola**.



Parabola

When both nappes of a double cone are intersected by a plane (not passing through the vertex), the cross section produces a **hyperbola**.



Hyperbola

1. Get a flashlight and shine a beam of light on a wall in a dark room. What happens to the light on the wall in the following situations?

   a. The beam is perpendicular to the wall.
   b. The beam is held at an oblique angle to the wall.
   c. The flashlight is moved closer to the wall.
   d. The flashlight is moved away from the wall.

2. Pour a coloured solution into a cone-shaped cup.

   a. Hold the cup so the central axis is vertical. What is the shape of the conic formed by the surface of the solution?

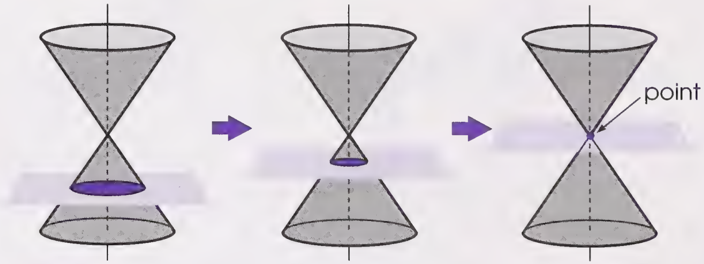
   b. Tilt the cup at an angle between 45° and 60°. What is the shape of the conic formed?

Check your answers by turning to the Appendix.

The shapes that are produced by the intersection of a plane and a double-napped cone are **conic sections**. The circle, ellipse, parabola, and hyperbola are called **proper** or **standard conic sections**. Other shapes are possible and are referred to as **degenerate conic sections**. The term *degenerate* refers to a change from the regular form.

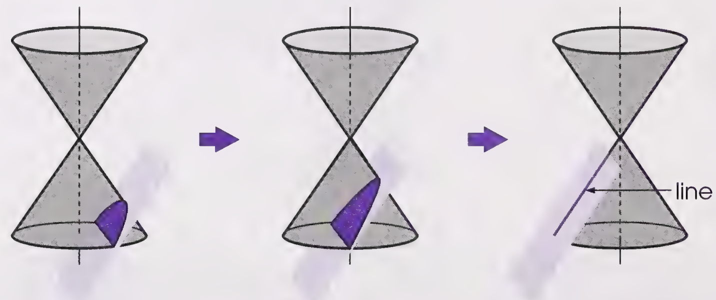
Examine the following degenerate cases.

Case 1: When a horizontal plane approaches the common vertex of a double-napped cone, the circle gets smaller. At the vertex, the circle becomes a **point**. You can say that the circle becomes a point-circle, or a circle of radius zero.

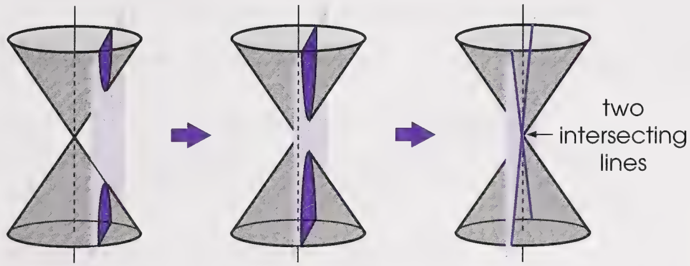


The point-circle is an example of a **degenerate conic**.

Case 2: When a plane parallel to a generator approaches the common vertex of a double-napped cone, the parabola first gets larger and then becomes narrower. At the vertex, the parabola ultimately becomes a **straight line**.



Case 3: When a vertical plane approaches the axis of symmetry of a double-napped cone, the hyperbola gets larger. At the axis of symmetry, the hyperbola becomes **two straight lines crossing at the vertex**.



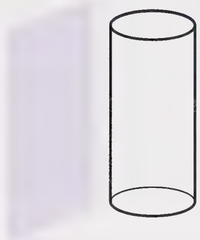
The cutting plane through the vertex produces a pair of intersecting lines as soon as the angle of the cutting plane exceeds the angle of the generator.

What happens when a plane is rotated from less than 90° to the axis, to the horizontal position? An ellipse changes to a circle.

Other degenerate conics are **two parallel lines** and the case of **no locus**. These two degenerate cases can be observed if you look at a cylinder. A cylinder (such as a pipe) can be considered as an extreme case of the cone.

If the plane is parallel to the sides of the cylinder and it touches the cylinder, it will produce **one line**.

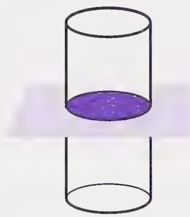
If the plane is parallel to the sides of the cylinder and cuts into the cylinder, it will produce **two parallel lines**.



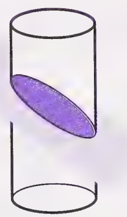
If the plane is parallel to the sides of the cylinder but doesn't touch the cylinder at all, it will produce **no curve** of intersection. This is referred to as the case of no locus.

If you take a cylinder and cut it with a plane in various ways, you get examples of conics.

If the plane cuts the cylinder at right angles, you will get a circle.



If the plane cuts the cylinder at any angle less than 90°, an **ellipse** is produced.



3. Take some clay or plasticine and roll it into a cylinder. Make the following cuts and state what you observe at the intersection of each cut.

   a. Slice the cylinder perpendicular to the vertical axis.

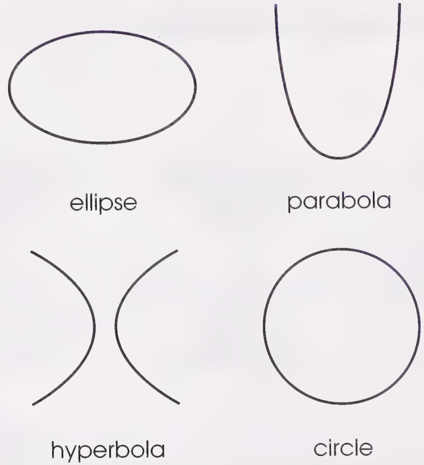
   b. Slice the cylinder at an angle to the axis that is less than 90° and greater than 0°.

Check your answers by turning to the Appendix.

Summary of Conic Shapes

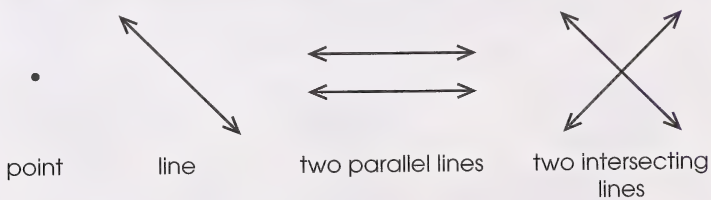
The primary conic sections are the ellipse, the parabola, the hyperbola, and the circle. The circle is a special type of ellipse. Therefore, a circle is a limiting case of an ellipse.

Primary, proper, and standard mean the same thing.



Summary of Degenerate Conics

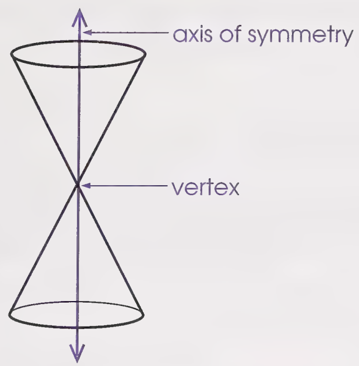
The degenerate conic sections are the point, the line, two parallel lines, two intersecting lines, and the case of no locus.



Example

Draw a diagram of a double-napped cone with a vertical axis of symmetry.

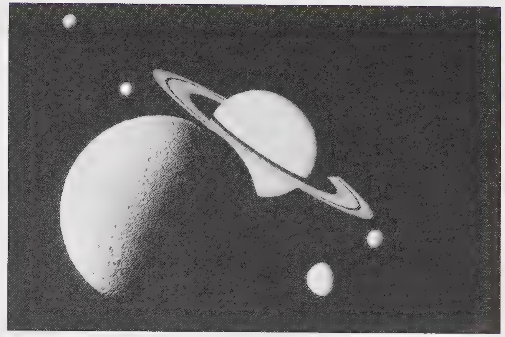
Solution



4. Make a list of ways you can physically produce each of the following shapes:

a. a circle
b. a parabola
c. an ellipse
d. a hyperbola

Check your answers by turning to the Appendix.

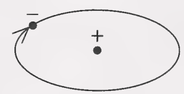


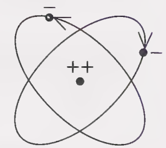
Scientists have discovered some amazing ties between the conic sections and astronomy. They have discovered planets and other celestial bodies move about in space following paths that are conic sections. In other words, some planets or comets follow the path of a circle or an ellipse. There are bodies that cross the heavens in straight lines. There are others that come into the solar system from deep space; they then turn around and head back into deep space in paths that resemble parabolas and hyperbolas.

Can you think of any conic sections that are a part of everyday life?

- **Elliptical and Circle Cases**

The paths of electrons moving about in an atom are examples of circles and ellipses.





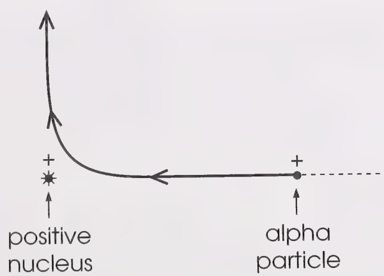
There are certain shapes in nature that are very close to being elliptical in shape. Examples of these are the eyeball of a mammal, the egg of a reptile, certain types of watermelon, raindrops at certain elevations, and so on.

- **Parabolic Cases**

When you play volleyball, you hit the ball upwards. The path it follows up and then down maps out the path of a parabola. When you squirt water upwards with a garden hose, the path of the water is also a parabola. You can probably think of many more examples.

- **Hyperbolic Cases**

When a positively charged alpha particle is repelled by a positively charged nucleus, the path the particle follows is hyperbolic.



Get some moist clay or plasticine and mold it into the shape of a right-circular cone. Use a sharp blade (knife or razor) to make a horizontal cut, a vertical cut, and a slant cut. What do you observe for each cut?

Cut two cones from styrofoam or any other similar material. Make sure the two cones are of the same size. Attach the two cones at their vertices so that they form a vertical double cone. Make cuts through the cone, not passing through the vertex. What do you observe at the cross section?

5. Describe the shape that would occur if the following experiments were performed.

   a. Pour a litre of water into an empty car tire that is sitting upright.

   b. Cut a cylindrical pipe at an angle of 45°, and view the cross section.

   c. Cut a pipe at various angles, and view the cross sections.

6. Take a cone-shaped cup, but do not flatten it. Cut off a slice parallel to the generator. (You are cutting through only one layer of cup surface.) What do you see?

7. **a.** A circle is the limiting case of the ellipse. Explain.

   b. If a circle is drawn on a thin rubbery sheet, how could you transform it into an ellipse?

View the video titled *Cutting the Curves* from the *Discovering Conics* series, ACCESS Network. This video shows how the various conics, including the degenerate forms, can be produced by cutting a cone and a cylinder with a plane. This video is available from the Learning Resources Distributing Centre.

8. What conditions are necessary to produce the conic sections using a cone and an intersecting plane?

Check your answers by turning to the Appendix.

The conic sections are part of the many geometric patterns that occur in the physical world. In this activity you investigated the physical properties of the four conics and discovered the degenerate cases.

Activity 2: Effects of Changing the Numerical Coefficients of the General Quadratic Equation

In order to study the effects of changing the values of A, C, D, E, and F in the general quadratic equation

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $B = 0$, you could change these values one at a time, two at a time, or up to all five at a time while the others are held constant.

Changing each of these values or **parameters** requires trying several values for each. This means you would need to graph at least three or four equations for each change in parameter.

This takes a lot of time! However, with the advent of graphing calculators and computer graphing programs, graphing a large number of equations is fairly quick and easy.

Calculator sequences in this course are shown for the CASIO *fx-7700G*. If you have a graphing calculator on loan from the Alberta Distance Learning Centre, it should be preprogrammed; you may omit the section on programming. If you are using a different calculator, consult your owner's manual for instructions.

Note: If you are using the TI-81 graphing calculator, a sample program is given in the Appendix.

Zap-a-Graph™ is the suggested graphing program (available for the Macintosh). You may be able to obtain this program from your school or local library.

Whichever graphing aid you choose, consult the manual on how to graph equations. It is easiest to use a program which allows you to just enter the values for the parameters.

The following is a sample calculator program for the CASIO *fx-7700G* graphing calculator.

This sample program includes the instructions for entering the program and the actual program you enter. The instructions explain exactly which keys to press to get a particular symbol. Notice that you may have to press several keys to get one letter or symbol. Follow the instructions carefully when entering the program. Follow the program line by line to be sure you are entering it exactly as given.

Sample Program for the CASIO *fx-7700G*

Read the keystroke chart and the sample program very carefully before you begin programming your calculator. Then return to this point and start entering the program.

Note: If at some point while entering the program you leave the calculator on long enough that it automatically shuts off, just turn the calculator on and return to your program by pressing MODE 2 ▼ (to your program number) EXE.

The following keystroke chart shows the keystrokes involved in obtaining certain characters on the CASIO *fx-7700G* graphing calculator.

Symbol	Keystrokes
"	ALPHA F2
=	SHIFT Range F2 F1
?	SHIFT Range F4
:	SHIFT Range F6
⇒	SHIFT Range F1 F1
Graph Y=	Graph
exponent 2	SHIFT √
◢	SHIFT Range F5

To begin programming, turn the calculator on and press [MODE] [2] [▼] (to an empty program number) [EXE].

To enter the sample program, do these steps:

Step 1: Enter the word CONICS. To get letters, use [ALPHA] followed by the letter. For example, the letter *C* is obtained by pressing [ALPHA] [ln].

[ALPHA] [ln] [ALPHA] [9] [ALPHA] [8] [ALPHA] [(]
[ALPHA] [ln] [ALPHA] [×] [EXE]

```
CONICS
```

Note: You may edit the program by using the arrow keys (to move up, down, right, and left) and [DEL] to delete any entry or [SHIFT] [DEL] (INS) to insert a missing entry.

Step 2: Enter the range. The standard range settings of $x = -10$ to 10 and $y = -10$ to 10 will not give a perfect circle on the calculator, since the scales are not the same on both axes. To set the same scale on both axes, make $x = -14.1$ to 14.1 and $y = -9.3$ to 9.3.

[Range] [–] [1] [4] [•] [1] [SHIFT] [→] [1] [4] [•] [1]
[SHIFT] [→] [1] [SHIFT] [→] [–] [9] [•] [3] [SHIFT]
[→] [9] [•] [3] [SHIFT] [→] [1] [EXE]

```
CONICS
Range –14.1, 14.1
, 1, –9.3, 9.3, 1
```

Note: In the CASIO *fx-7700G* graphing calculator you may use the subtraction key [–] for both the negative symbol and the minus sign. However, do not use the negative key [(–)] for subtraction.

Step 3: Using the keystrokes chart, enter the following program.

(Remember, when entering a letter, press **ALPHA**, followed by the letter.) Your screen will scroll upwards. You may find it helpful to check your entries after each line.

$"A="? \rightarrow A:"B="? \rightarrow B:"C="? \rightarrow C$

(Then press **EXE**.)

```
CONICS
Range −14.1, 14.1
, 1, −9.3, 9.3, 1
"A = "?→A:"B = "?→B:
"C = "?→C
```

$"D="? \rightarrow D:"E="? \rightarrow E:"F="? \rightarrow F$

(Then press **EXE**.)

```
CONICS
Range −14.1, 14.1
, 1, −9.3, 9.3, 1
"A = "?→A:"B = "?→B:
"C = "?→C
"D = "?→D:"E = "?→E:
"F = "?→F
```

$C=0 \Rightarrow \text{Graph } Y=\left(-1\times A\times X^2-D\times X-F\right)$

$\div(B\times X+E)$ ◢

```
"C = "?→C
"D = "?→D:"E = "?→E:
"F = "?→F
C = 0⇒Graph Y = (−1×
A×X²−D×X−F)÷(B×
X + E) ◢
```

Graph $Y = \left(-1 \times (B \times X + E) + \sqrt{\ }\left((B \times X + E)^2 - 4 \times C \times \left(A \times X^2 + D \times X + F\right)\right)\right)$

$\div (2 \times C)$

(Then press **EXE**.)

```
AxX²–DxX–F)÷(Bx
X + E) ◢
Graph Y = (–1(BxX +
E) + √((BxX + E)²–4x
Cx(AxX² + DxX + F)))
÷(2xC)
```

Graph $Y = \left(-1 \times (B \times X + E) - \sqrt{\ }\left((B \times X + E)^2 - 4 \times C \times \left(A \times X^2 + D \times X + F\right)\right)\right)$

$\div (2 \times C)$ ◢

```
Cx(AxX² + DxX + F)))
÷(2xC)
Graph Y = (–1(BxX +
E)–√((BxX + E)²–
4xCx
(AxX² + DxX + F)))÷(
2xC) ◢
```

Step 4: When programming is completed, press MODE 1 AC to get out of the programming mode.

Step 5: To run this program, press the following:

SHIFT Range F3 (the program number) EXE

To clear the screen, press SHIFT F5 EXE.

Now you can enter your values of A, B, C, D, E, and F.

Press a number for A; then press EXE. Press $B = 0$; then press EXE. Press a number for C; then press EXE. Do this until you have entered all six values.

Graphing Equations

Now you are ready to try graphing some equations. Carefully follow the instructions at the end of each sample program. You may have to refer to your manual for help on how to set the range and use the zoom features of the calculator.

In this activity you will be given a number of equations to graph. The resulting graph is given immediately after each equation. Graph each equation with whatever graphing technology you are using and compare it to the given graph. The purpose of this activity is to help you familiarize yourself with using graphing technology and to give you confidence that you are obtaining the correct graphs.

Note: It is recommended that you use a graphing calculator rather than a computer program in order to become more familiar with the calculator. The reason is that graphing calculators are permitted while writing the Mathematics 30 Final Test and the Mathematics 30 Diploma Examination.

If you are already familiar with the graphing calculator or computer program you are using, you may omit the examples and go on to the questions.

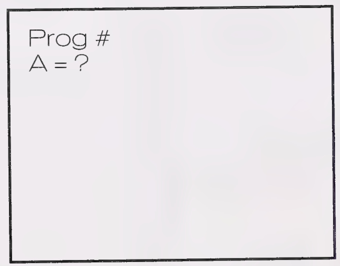
Example 1

Graph the equation $x^2 + y^2 - 4x + 6y = 0$ using a graphing calculator and compare your graph with the one that is given.

Solution

Step 1: Run the conics program.

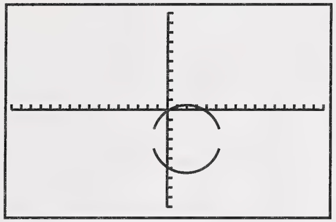
SHIFT Range F3 (the program number) EXE



Step 2: Enter the values of A, B, C, D, E, and F as the program requests. Press EXE after each value entered.

1 EXE 0 EXE 1 EXE

− 4 EXE 6 EXE 0 EXE



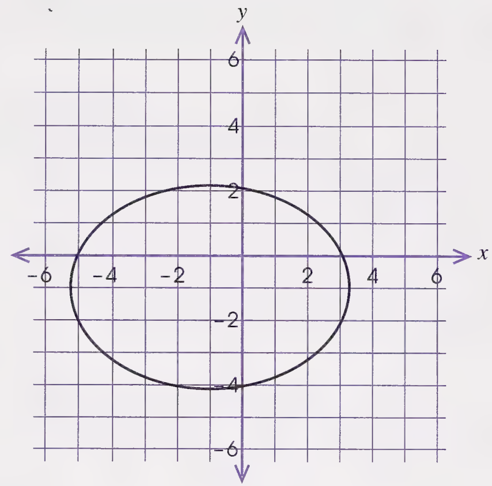
Note: The breaks on each side of the circle are due to the way the graphing calculator makes the graph. The graphing calculator makes the graph by graphing the upper half and lower half separately, resulting in a split graph.

If your graph does not resemble the given graph, try entering the values again. Be sure you enter a zero value for each missing parameter.

Example 2

Graph the equation $x^2 + 2y^2 + 2x + 4y - 16 = 0$.

Solution

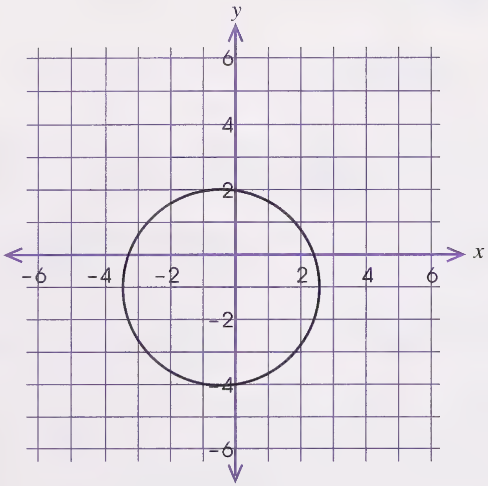


Compare your graph to the given graph.

Example 3

Graph $2x^2 + 2y^2 + 2x + 4y - 16 = 0$.

Solution

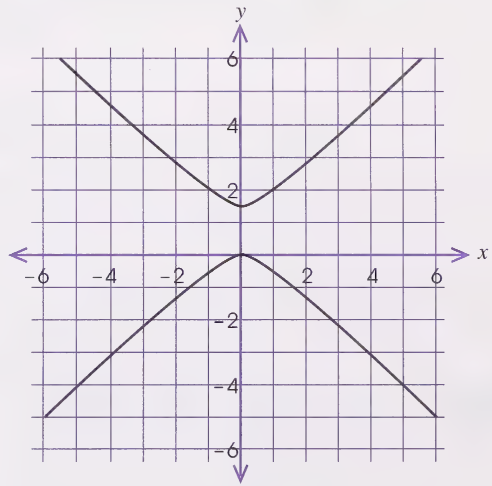


Compare your graph to the given graph.

Example 4

Graph $2x^2 - 2y^2 + 3y = 0$.

Solution

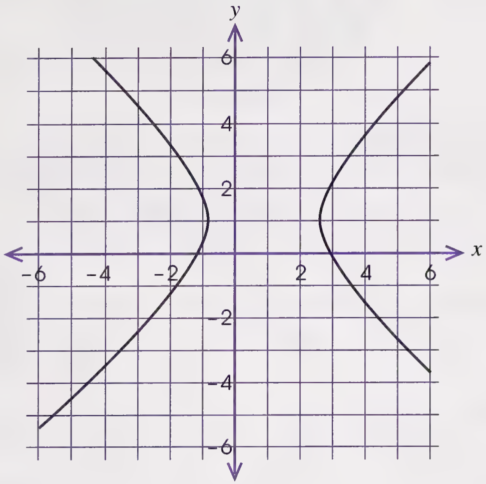


Compare your graph to the given graph.

Example 5

Graph $2x^2 - 2y^2 - 4x + 3y - 5 = 0$.

Solution



Compare your graph to the given graph.

Use a graphing calculator (or computer program) to answer the following questions.

1. a. Graph $x^2 + y^2 + 4x + 8y - 10 = 0$.
   b. What type of curve is created?
   c. What are the values of the coefficients of the curve?

2. a. Graph $4x^2 + y^2 + 4x + 8y - 3 = 0$.
   b. What type of curve is created?
   c. What are the values of the coefficients of the curve?

3. a. Graph $x^2 + y^2 + 4x - 8y - 16 = 0$.
   b. What type of curve is created?
   c. What are the values of the coefficients of the curve?

4. a. Graph $x^2 + 4x + 6y - 1 = 0$.
   b. What type of curve is created?
   c. What are the values of the coefficients of the curve?

Check your answers by turning to the Appendix.

Now that you have tried graphing various equations, you should be ready to use a graphing calculator or computer program to study the effects that changing the parameters has on the graph. You will discover the effects that each variable produces on the graphs through investigations.

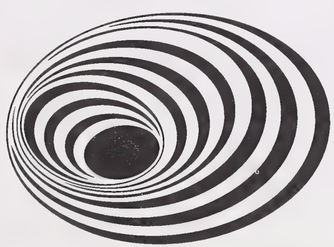
Use a graphing calculator (or computer program) to answer question 5.

5. Investigate the equation $x^2 + y^2 + 2x + 2y - 16 = 0$ by systematically changing each of the coefficients A, C, D, E, and F to -10, -2, 0, 2, and 9. Graph the relation and answer the following questions.

   a. What effect does A have on the curve when A changes?
   b. What effect does C have on the curve when C changes?
   c. What effect does D have on the curve when D changes?
   d. What effect does E have on the curve when E changes?
   e. What effect does F have on the curve when F changes?

Check your answers by turning to the Appendix.

In this activity you examined the different curves that are produced by changing the coefficients in the general quadratic equation. The next activity emphasizes the changes in two specific parameters.



Activity 3: Effects of Changing A and C in $Ax^2 + Cy^2 + Dx + Ey + F = 0$

Now you will begin your investigations of the specific changes that occur in the general conic when $B = 0$. The general form of the equation that you will be using is $Ax^2 + Cy^2 + Dx + Ey + F = 0$. What shapes are possible when A and C are varied? To simplify the observations, D and E will be made equal to zero where possible.

Use a graphing calculator to answer the following questions. You may use a computer program if you wish.

You will be required to pick values for one or two of the parameters. Keep the values you pick to within ± 10, unless you are specifically asked to use very large values or very small values. Graph the relations as directed and answer the questions.

1. Use the equation $Ax^2 + Cy^2 - 16 = 0$ to investigate the shape of the curve when $A = C$ and $B = 0$. Use various positive and negative values for A and C. Try some large values, like 100 or 1000, as well.

   a. What is the resulting figure?
   b. What happens when A and C are large?
   c. What happens when A and C are negative?

2. What is the shape of the curve when either $A = 0$ or $C = 0$ in the equation $Ax^2 + Cy^2 + 4x + 8y - 16 = 0$? Try some positive and negative values for A when $C = 0$. Then try some positive and negative values for C when $A = 0$.

   a. What is the shape of the graph when either $A = 0$ or $C = 0$?

   b. If A is a positive value and $C = 0$, which way does the graph open?

   c. If A is a negative value and $C = 0$, which way does the graph open?

   d. If $A = 0$ and C is a positive value, which way does the graph open?

   e. If $A = 0$ and C is a negative value, which way does the graph open?

3. Investigate the shape of the curve when A and C are not equal and $B = 0$. Use both positive and negative values for A and C. Also, vary the size of A as compared to C. Use the equation $Ax^2 + Cy^2 - 16 = 0$.

   a. What are the resulting shapes when A and C have the same sign?

   b. What are the resulting shapes when A and C have different signs?

4. Investigate the effects on an ellipse when A and C are interchanged. Using $Ax^2 + Cy^2 - 16 = 0$, pick values for A and C where A is less than C. Then interchange the values of A and C. What can you conclude about the relative size of A and C and the orientation of the graph?

5. Investigate the effects on a hyperbola of interchanging A and C. Use $Ax^2 + Cy^2 - 16 = 0$. Pick values for A and C that will give a hyperbola. Now interchange the values of A and C. (Notice this involves a sign change.) Try just exchanging the signs. What can you conclude about the sign of A and C and the position of the graph?

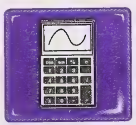
6. By referring to questions 1 through 5, summarize the types of shapes that are possible if $B = 0$, and A and C are varied. Explore the variations when A and C are equal, when either A or C is equal to zero, when A and C are not equal but have the same signs, and when A and C are not equal and have different signs.

Check your answers by turning to the Appendix.

Did you enjoy graphing the many quadratic equations? You should be able to handle a graphing calculator or graphing program with ease after this activity.

Activity 4: Effects of Changing A or C in $Ax^2 + Cy^2 + Dx + Ey + F = 0$

In the previous activity you saw that if you varied A and C, you could get one of the four basic conics—a circle, an ellipse, a parabola or a hyperbola—depending on the values for A and C. You also saw that changing the size of A and C gave larger or smaller circles. In this activity you will vary A or C one at a time and see how the different values affect the ellipse, the parabola, and the hyperbola.



Use a graphing calculator to answer questions 1 to 3. Graph the equations as instructed and answer the questions. You may use a computer program if you wish.

1. Investigate the effect of changing the value of A in the equation of a parabola, $Ax^2 + 4x - 2y - 8 = 0$. Try some positive values and some negative values for A.

  a. How does the direction the graph opens change as the value of A goes from negative to positive?

  b. How does the shape of the graph change as the value of A becomes larger?

  c. What can you conclude about the vertex of the parabola as A changes?

Notice that the graph opens down when A is negative and up when A is positive. This is opposite to what you saw in question 2 of Activity 3. This is because the sign of E is negative in $Ax^2 + 4x - 2y - 8 = 0$ while E was positive in the equation $Ax^2 + Cy^2 + 4x + 8y - 16 = 0$.

2. Using $Ax^2 + 7y^2 - 36 = 0$, vary the size of A. Use $A = 1$, $A = 3$, and $A = 5$.

  a. What happens to the shape of the ellipse as the value of A is changed from 1 to 3 to 5? (**Note:** Here, A is less than C.)

  b. What can you conclude about the shape of the ellipse when the value of A gets closer to the value of C?

  c. What value would you use for A to make this ellipse into a circle?

  d. What do you think the effect will be if you make $A = 7$ and vary the value of C? If you are unsure, test the values of 1, 3, and 5 for C. (Note that C is less than A here.)

Notice that both the horizontal and vertical ellipses approach a circle as the values of A and C approach the same value. When the values of A and C are the same, you have a circle. You can say that an ellipse changes into a circle when the values of A and C become equal.

3. Using $Ax^2 - 4y^2 - 16 = 0$, vary the size of A to investigate the effects on a hyperbola of changing the size of A. Use $A = 1$, $A = 4$, and $A = 6$.

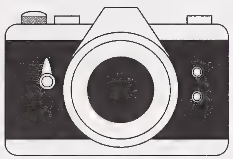
   a. What happens to the shape of the hyperbola as the value of A is changed from 1 to 4 to 6?

   b. Is the effect the same if you let $A = -4$ and use the values of 1, 4, and 6 for C? If you are unsure, test the values of 1, 4, and 6 for C in the equation $-4x^2 + Cy^2 - 16 = 0$.

4. Summarize the effects of changing A or C when you have the equations of a parabola, an ellipse, and a hyperbola. Refer to questions 1 to 3 to help you complete your summary.

Check your answers by turning to the Appendix.

Using camera tricks you can compress or stretch a figure. Similarly, you can alter the shape of a parabola, ellipse, or hyperbola by changing the coefficients A and C in the general equation.

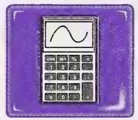


Activity 5: Effects of Making *D* or *E* Equal to Zero in $Ax^2 + Cy^2 + Dx + Ey + F = 0$

In this activity you will try to find the values that put the vertex of a parabola or the centres of a circle, an ellipse, and a hyperbola at the origin of the coordinate plane.

Vertex of a Parabola

First, you will find the values that put the vertex of a parabola at the origin. As you already know from question 2 of Activity 3, a parabola has either A or C equal to zero.



Use a graphing calculator (or computer program) to answer questions 1 to 5. Graph each equation as instructed and answer the questions.

1. Using $7x^2 + Dx + 3y = 0$, choose various values for D. What value of D puts the vertex of the parabola at the origin?

2. Using $5y^2 + 3x + Ey = 0$, choose various values for E. What value of E puts the vertex of the parabola at the origin?

Check your answers by turning to the Appendix.

Centres of a Circle, an Ellipse, and a Hyperbola

What values will put the centres of a circle, an ellipse, and a hyperbola at the origin?

3. The equation $3x^2 + 3y^2 + Dx + Ey - 81 = 0$ represents a circle. Let $D = E$ and choose various values for D and E. What values of D and E put the centre of a circle at the origin?

4. Let $D = E$ in the equation $2x^2 + 7y^2 + Dx + Ey - 49 = 0$. Graph the equation by choosing various values for D and E.

- **a.** What values for D and E put the centre of an ellipse at the origin?
- **b.** Try other values for A and C which will make an ellipse. Do equal values of D and E put the ellipse at the origin?

5. Let $D = E$ in the equation $2x^2 - 9y^2 + Dx + Ey - 25 = 0$. Choose various values for D and E.

- **a.** What values of D and E put the centre of the hyperbola at the origin?
- **b.** Exchange the value of A and C. Let $D = E$ and try various values for D and E. Do the same values of D and E put the centre of the hyperbola at the origin?
- **c.** Try other values for A and C which will make a hyperbola. Do the same values for D and E put the hyperbola at the origin?

6. Give a summary of the values of D and E that put all of the curve centres at the origin.

Check your answers by turning to the Appendix.

In this activity you discovered that when $D = E$, only one value (0) centres each of the conic sections at the origin.

Activity 6: Effects of Changing *D*, *E*, or *F* in $Ax^2 + Cy^2 + Dx + Ey + F = 0$

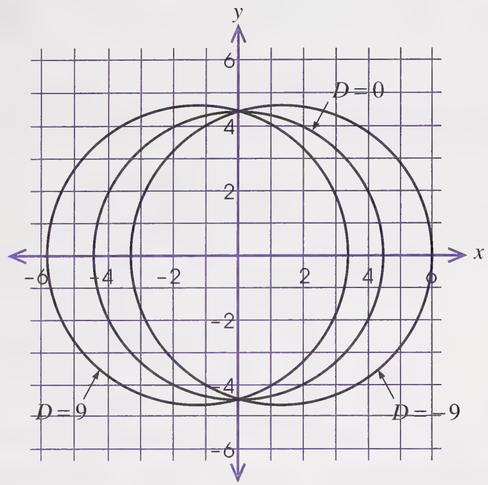
In Activity 5 you saw that if you made D and E equal to zero, you could produce a circle, an ellipse, a parabola, or a hyperbola at the origin. In this activity you will look at the effects of D and E when they are not equal to zero.

First, you will look at the effects of changing D on each of the curves.

Example 1

Graph the circle equation $4x^2 + 4y^2 - 81 = 0$. Let $D = 9$, $D = 0$, and $D = -9$. Describe the effects of changing D.

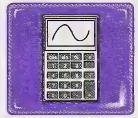
Solution



What is the effect when $D = 9$? The circle moves to the left when D increases.

What is the effect when $D = -9$? The circle moves to the right when D decreases.

What will be the effect of other values of D? You will get the same results. When D increases, the circle moves to the left; when D decreases, the circle moves to the right.



Use a graphing calculator (or computer program) to answer questions 1 to 4. Graph each equation as instructed and answer the questions.

1. Graph the equation $x^2 + 4y^2 - 16 = 0$ which represents an ellipse centred at the origin. Graph the results when $D = 5$ and $D = -5$.

   a. Which way does the graph move when D increases?

   b. Which way does the graph move when D decreases?

   c. Interchange A and C in the equation. Let $D = 5$ and $D = -5$ again. Is the effect the same?

2. When $D = 0$ in the equation $3x^2 + Dx + 9y = 0$, the graph of the parabola has its vertex at the origin. Let $D = 7$, $D = 0$, and $D = -7$; and graph the results.

   a. How does the position of the parabola change when $D = 7$ as compared to the position when $D = 0$?

   b. What is the effect on the position of the graph when $D = -7$ as compared to the position when $D = 0$?

   c. Graph other positive and negative values for D. Is the effect the same?

3. a. Graph the equation $x^2 - 4y^2 - 16 = 0$. Describe the shape and position of the graph. Graph the equation when $D = 5$ and $D = -5$.

b. What is the effect when $D = 5$?

c. What is the effect when $D = -5$?

d. Graph other positive and negative values for D. Is the effect the same?

4. a. Graph the equation $-x^2 + 4y^2 - 16 = 0$. What type of curve is created and on which axis does it lie? Graph the equation when $D = 5$ and $D = -5$.

b. What is the effect when $D = 5$?

c. What is the effect when $D = -5$?

d. Are the changes here the same as the changes that occurred with the hyperbola on the x-axis? Explain.

5. Summarize the movements that occur with each type of curve when D increases and when D decreases.

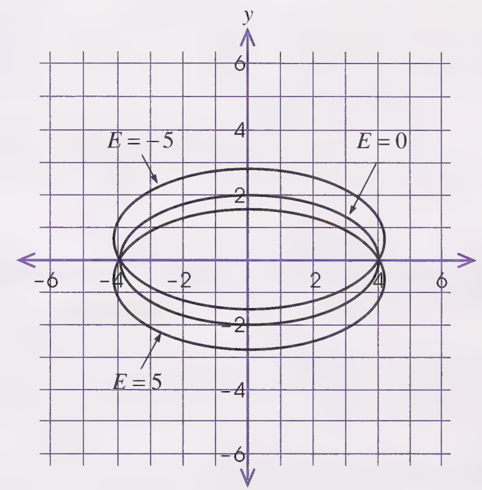
Check your answers by turning to the Appendix.

You will now look at the effects of changing E on the graphs of a circle, an ellipse, a parabola, and a hyperbola.

Example 2

Graph the equation $x^2 + 4y^2 - 16 = 0$. Let $E = 5$ and $E = -5$, and graph the results. Describe the effects of changing E.

Solution



The graph of $x^2 + 4y^2 - 16 = 0$ is an ellipse centred at the origin. Notice that the ellipse moves down when E is positive, and up when E is negative. The effect is the same for other values of E.

Use a graphing calculator (or computer program) to answer questions 6 to 9. Graph each equation as instructed and answer the questions.

6. Graph the equation $3x^2 + 3y^2 - 49 = 0$. Let $E = 8$ and $E = -8$, and graph the results.

   a. What is the effect when $E = 8$?

   b. What is the effect when $E = -8$?

   c. Graph other positive and negative values for E. Is the effect the same?

Note: Although changing the size of E increases the size of the ellipse, the dominant effect is to move the ellipse up or down.

7. Graph the equation $y^2 + 6x + Ey = 0$ when $E = 0$. Let $E = 4$ and $E = -4$, and graph the results.

   a. How does the position of the graph change when $E = 4$ as compared to the position when $E = 0$? Why?

   b. How is the position of the graph affected when $E = -4$ as compared to the position when $E = 4$? Why?

   c. Graph other positive and negative values for E. Is the effect the same?

8. Graph the equation $x^2 - 4y^2 - 16 = 0$. Let $E = 10$ and $E = -10$, and graph each result.

   a. Describe the effects on the graph when E is positive and when E is negative.

   b. Do you get the same result for other positive and negative values of E?

9. **a.** If the equation in question 8 is changed to $-x^2 + 4y^2 - 16 = 0$, and E is a positive or negative value, what changes will you expect to see as compared to the graphs in question 8?

   b. How does the movement of a hyperbola on the y-axis compare with the movement of a hyperbola on the x-axis when the value of E is changed?

10. Summarize the effects of E on the various curves.

Check your answers by turning to the Appendix.

In the next set of questions you will look at the effects of changing F on the graphs of a circle, an ellipse, a parabola, and a hyperbola.

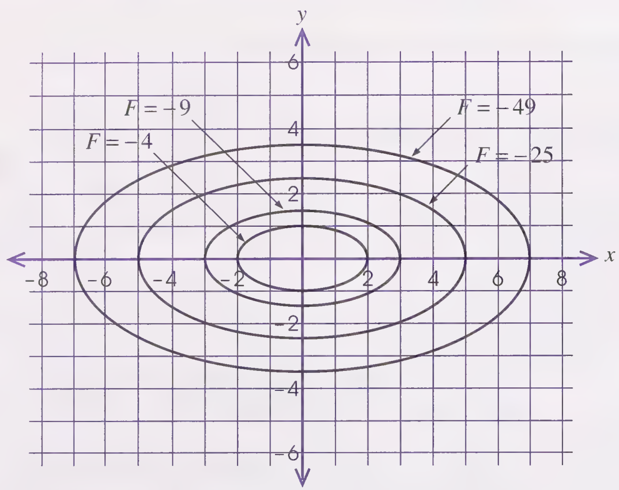
Example 3

Graph the equation $x^2 + 4y^2 - 4 = 0$. Let $F = -9$, $F = -25$, and $F = -49$; then graph each result. Use the graphs (and any additional graph you may have plotted yourself) to answer the following:

- What happens to the graph when F has various negative values?

- Interchange A and C. Let $F = -9$, $F = -25$, and $F = -49$; then draw the graphs. (These graphs will not be given in the solution.) Is the effect the same?

Solution



Using various negative values, it is evident that the ellipse increases in length as F decreases.

When A and C are interchanged, the effect of the graphs is the same except that the graphs are vertical ellipses.

Try graphing the equation $x^2 + 4y^2 + 4 = 0$; let $F = 9$, $F = 25$, and $F = 49$. What do you notice? You should notice that there is no graph when $F = 4$, $F = 9$, $F = 25$, $F = 49$, or any other positive value of F. As in the case of a circle, you have two squared values equal to a negative value. This is impossible!

11. Graph the equation $2x^2 + 2y^2 - 25 = 0$; then graph the equations when $F = -36$, $F = -49$, and $F = -81$.

a. What happens to the graph as F decreases?

b. Let $F = -1$ and -0.5. Do these values of F give a graph?

c. Let $F = 64$. What happens?

d. Why did you get no graph in question 11.c.? (**Hint:** Rewrite the equation as $2x^2 + 2y^2 = -64$.)

12. Graph the equation $x^2 + 6y = 0$. Let $F = 9$ and $F = -9$; then graph the results.

a. What happens to the position of the graph when the value of F is changed from 0 to 9?

b. What happens to the position of the graph when the value of F is changed from 0 to -9?

c. Graph other positive and negative values for F. Is the effect the same?

13. Graph the equation $y^2 + 6x = 0$. Then graph the equations when $F = 7$ and $F = -7$.

a. What happens to the position of the graph when F is changed from 0 to 7?

b. What happens to the position of the graph when F is changed from 0 to -7?

c. Graph other positive and negative values for F. Is the effect the same?

14. Graph the equation $x^2 - 4y^2 - 1 = 0$. Graph the equation when $F = -16$, $F = -36$, and $F = -64$.

a. What is the effect on the graph as the value of F is changed from -16 to -36 to -64?

b. Now let $F = 16$. Graph the equation. What is the effect on the graph? Why? (**Hint:** Try multiplying the equation by -1.)

c. Let $F = 49$ and $F = 81$. Graph the equations. What is the effect of giving F larger positive values?

15. Graph the equation $-4x^2 + y^2 - 1 = 0$. Let $F = -16$, $F = -36$, and $F = -64$; then graph the results.

a. What is the effect on the graph as F is changed from -16 to -36 to -64?

b. Let $F = 16$. What is the effect on the graph? Why?

c. Let $F = 49$ and $F = 81$. What is the effect on the graph?

16. Let $F = 16$, $F = 0$, and $F = -16$ in the equation $x^2 - 9y^2 + F = 0$. Graph each result.

a. What is the effect as F goes from 16 to -16?

b. What happens when $F = 0$? Explain, algebraically, why this happens.

c. Relate what happens when $F = 0$ to the cutting of a double cone by a plane.

17. Summarize the effects on a circle, an ellipse, a parabola, and a hyperbola when F is changed.

Check your answers by turning to the Appendix.

This ends your investigations on the effects on the various curves when D, E, and F are changed. Although there have been a number of different changes, in particular with the F value, you can make some general conclusions about the effects of changing D, E, and F.

View the video titled *Graphing the Curves* from the *Discovering Conics* series. This video shows how you can generate the different conics when $B = 0$. Stop the program before the start of the part showing $B \neq 0$. This video is available from the Learning Resources Distributing Centre.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

The intersection of a plane and a double-napped cone results in either a **standard conic section** or a **degenerate conic section**.

The standard conic sections are a circle, an ellipse, a parabola, and a hyperbola. The point, the line, two parallel lines, two intersecting lines, and the case of no locus are the degenerate conic sections. Some of these shapes can be observed in real-life situations.

1. What shape(s) does a spotlight form on the ice when used in skating shows? Would you classify the shape(s) as standard or degenerate?

2. When a baseball is hit or thrown, what is the shape of the baseball's path? Would you classify the shape as standard or degenerate?

3. Some comets travel along paths which appear once near the Sun and do not return. Why might this happen? (Relate your answer only in terms of a path.)

Check your answers by turning to the Appendix.

Changing the coefficient values A, C, D, E, and F in the general quadratic equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $B = 0$, affects the curves in various ways.

Changing the values of A and C may change the type of curve as well as the shape of the curve. As A changes from negative to positive, the curve changes from a hyperbola $(-A)$ to a parabola $(A = 0)$ to a circle $(A = 1)$. As C changes from negative to positive, the curve changes from a hyperbola to a circle $(C = 1)$ to an ellipse.

Changes in the coefficients D, E, and F also affect the curves in various ways.

In general, D moves the graph left or right. In the case of a circle and an ellipse, increasing D moves the curves left and decreasing D moves the curves right. In the case of a parabola and a hyperbola, D moves the curves left or right; the direction in which the curve moves depends on the sign of A or C as well as the sign of D.

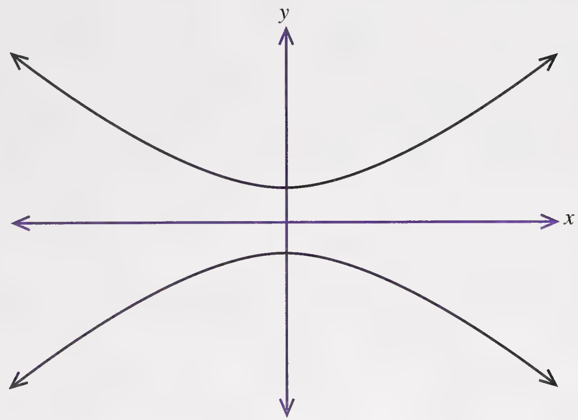
In general, E moves the graph up or down. In the case of a circle and an ellipse, increasing E moves the curve down and decreasing E moves the curve up. In the case of a parabola and a hyperbola, E moves the curves up or down; the direction in which the curve moves depends on the sign of A or C as well as the sign of E.

When F changes, the following changes may occur in the graph:

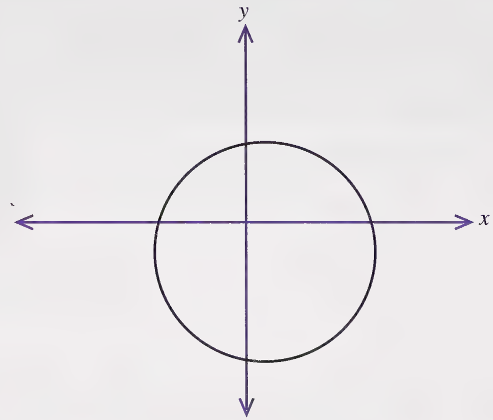
- Circle and Ellipse: Changing the value of F will change the size of the circle or ellipse or it will result in the degenerate case.
- Parabola: When the graph is a horizontal parabola, changing the value of F will move the parabola horizontally. When the graph is a vertical parabola, changing the value of F will move the parabola vertically.
- Hyperbola: Changing the value of F will cause the branches of the hyperbola to come together or move apart, or it could result in the degenerate case.

4. Give a possible combination of coefficients which would determine the following curves.

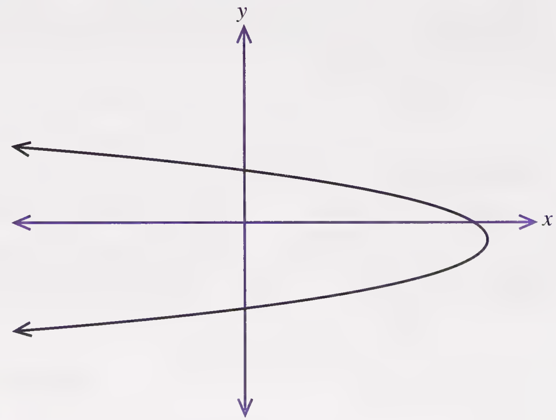
a.



b.



c.



Check your answers by turning to the Appendix.

Enrichment

In this section, you investigated the effects of changing the numerical coefficients in the equation
$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, when $B = 0$.

In your investigations you observed these effects on the positions of a circle, an ellipse, a parabola, and a hyperbola on the axes. Now you will look at some of the special effects that changing the value of *B* can have on the curves.

1. Graph the equation $x^2 + Bxy + y^2 - 25 = 0$ when $B = 1$ and $B = -1$. What effect does *B* have on the curve?

2. **a.** Let $B = 1$ and $B = -1$ in the equation $Bxy - 25 = 0$. Graph the results. What are the resulting curves?

b. What is the effect of changing *B* from positive to negative?

3. **a.** Graph the equation $x^2 + xy + 9y^2 - 9 = 0$. What type of graph do you get?

b. Let $B = 4$ and graph the equation. What changes do you see?

c. What type of graph do you get when $B = 7$?

Check your answers by turning to the Appendix.

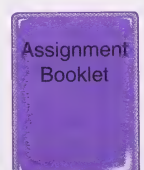
Complete viewing the video titled *Graphing the Curves* from the *Discovering Conics* series, ACCESS Network. This segment shows how you can generate some conics when $B \neq 0$. This video is available from the Learning Resources Distributing Centre.

Conclusion

In this section, you investigated how the circle, ellipse, parabola, and hyperbola, and their degenerate forms may be obtained through the intersection of a double-napped cone and a cylinder by a plane. You also explored the equations of the conic sections, describing how changing the coefficients of these equations affected the orientation, size, and shape of their corresponding graphs.

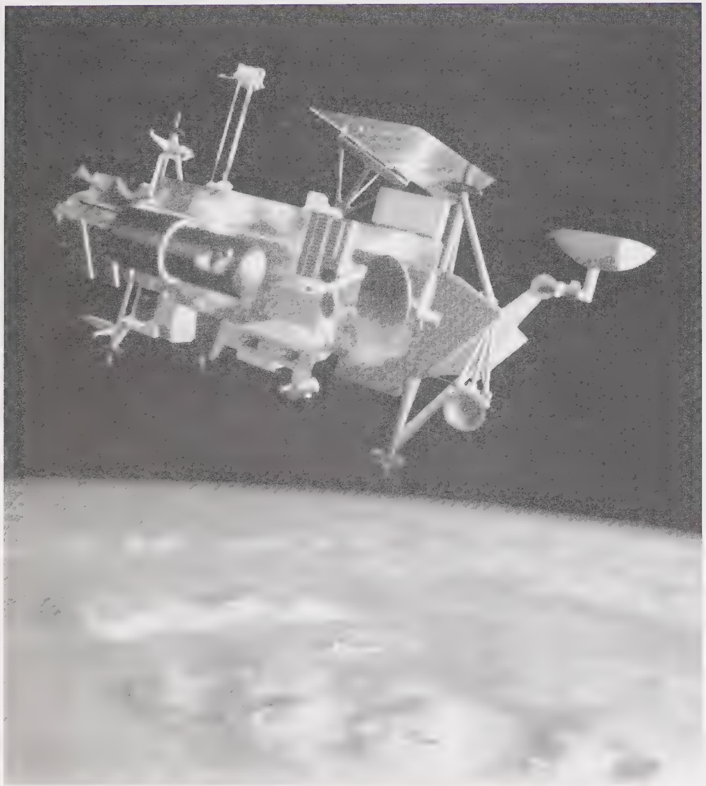
The conic sections have applications in a variety of physical situations. The orbits of the planets, satellites, and electrons are elliptical. The secret of a whispering gallery lies in its elliptical shape. Paths of sparks from fireworks, the trajectories of bullets, and the arcs formed by water droplets in a sprinkler are parabolic. The paths of some comets are hyperbolic. Can you find other applications in the sciences?

Assignment



You are now ready to complete the section assignment.

Section 2: Defining Conics



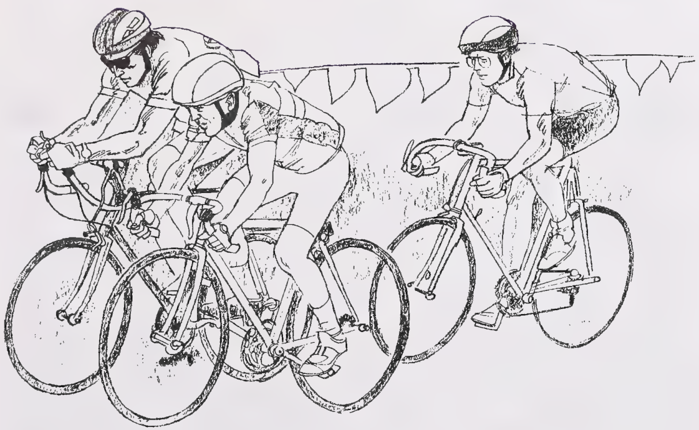
NASA

When you ride a bicycle, the hubs of the wheels trace a path that is parallel to ground. Communication satellites, as they orbit around Earth, maintain a fixed position relative to ground stations. A goat tied to a stake will graze in the circular area to which it is restricted.

In this section you will explore the conditions or restrictions that result in the conic sections—the circle, ellipse, parabola, and hyperbola. You will apply these locus definitions to derive the equations of these curves.

You will also define and develop the conic sections in terms of eccentricity—a ratio that involves distances from a fixed line and a fixed point.

Activity 1: Describing Conics as a Locus of Points



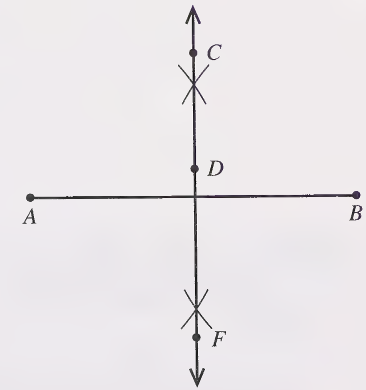
The paths of points can be described by some rule that defines all points on the curve. The following example will show you how to establish this rule.

Example 1

Choose two points, A and B, on a sheet of paper. Find all of the points that are equidistant from these two points.

Solution

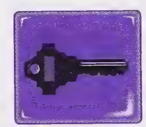
Draw a line segment AB. Using a compass, construct the perpendicular bisector of $\overline{AB}$. Locate points C, D, and F at random on this line.



$$d(CA) = d(CB)$$
$$d(DA) = d(DB)$$
$$d(FA) = d(FB)$$

You will notice that the points on the perpendicular bisector of AB are equidistant from point A and point B.

You can write a rule for all the points on this line.



A locus is a path or a set of points that follows a rule. This rule is called the **locus definition**. A locus definition for this example is as follows: The perpendicular bisector of $\overline{AB}$ is the locus of all points equidistant from two fixed points A and B.

Each conic has its own locus definition. You will discover these definitions through investigations on the different conics.

1. Draw a circle with a radius of 4 cm and centre C. Write a rule that will define all points on the curve; then give the locus definition for the set of points on the curve.

2. Using a compass and a sheet of ordinary lined paper, construct a figure by completing the following steps.

Step 1: At the bottom of the paper draw a line D.

Step 2: Label the lines above D as 1, 2, 3, and so on. The space between two adjacent lines on the paper measures 1 unit.

Step 3: On line 2 mark any point F.

Step 4: On line 1 label point V, 1 unit from D and 1 unit from F. (V is the point of intersection of line 1 and the perpendicular line segment from F to D.)

Step 5: With the compass point on F and a radius of 2 units from D, mark two points on line 2. With the compass point on F and a radius of 3 units from D, mark two points on line 3. **Remember:** Three spaces between the lines will give you 3 units of measure.

Continue this process until you get a reasonable number of points so that you can see the shape of the curve.

Step 6: Join the points with a smooth curve. What shape results?

Step 7: Choose a point P_1 on the curve. Measure $d\left(P_1F\right)$ and $d\left(P_1D\right)$. How do the distances compare? Choose other points on the curve, and find their distances to F and D. What is your conclusion? Write a rule that will define all points on the curve.

Step 8: Identify the fixed point and fixed line.

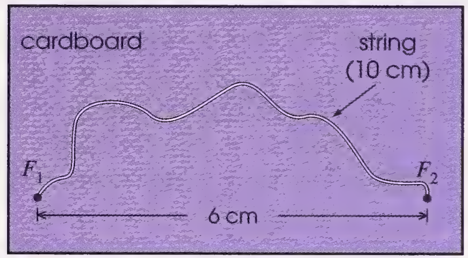
Step 9: Write a locus definition for the set of points on this curve.

The fixed point F is the **focus**, and the fixed line D is called the **directrix** of this curve.

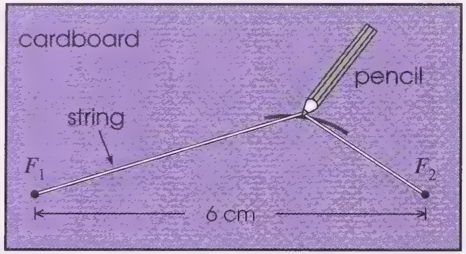
3. Find a piece of cardboard about 40 cm by 30 cm and a piece of string about 20 cm long; then complete the following steps:

Step 1: Make two holes about 6 cm apart in the cardboard. Label the holes F_1 and F_2.

Step 2: Pass an end of the string through each of the two holes and tie a large knot at each end so that the length of the string on the cardboard measures 10 cm.



Step 3: Catch the string with a pencil. Keeping the string taut, move the pencil to trace a curve.



a. What shape is the curve?

b. Choose points A, B, and C anywhere on the curve. Find the following measures with a ruler.

- $d(AF_1)+d(AF_2)$
- $d(BF_1)+d(BF_2)$
- $d(CF_1)+d(CF_2)$

c. What is your conclusion?

d. Write a rule that will define all points on the curve. (The two fixed points F_1 and F_2 are called the **foci** of this curve.)

e. Give the locus definition for the set of points on this curve.

f. Change the length of the string. What effect does it have on the shape of the curve?

g. Move the holes closer to each other. What effect does it have on the shape of the curve?

h. If $F_1 = F_2$, what is the shape of the curve?

4. On a sheet of paper label two points F_1 and F_2, spacing them 8 cm apart. Use a compass to locate two points that are 4 cm from F_1 and 8 cm from F_2. (You will get a point on either side of line segment F_1F_2.)

Then locate two points that are 3 cm from F_1 and 7 cm from F_2, and two points that are 2 cm from F_1 and 6 cm from F_2. Then join all the points. Reverse the process in order to locate two points that are 8 cm from F_1 and 4 cm from F_2. Next locate points that are 7 cm from F_1 and 3 cm from F_2, and two points that are 6 cm from F_1 and 2 cm from F_2. Then join all the points.

a. What conic is created?

b. Choose points P_1, P_2, and P_3 on the curves. Find the following measures.

- $|d(P_1F_1)-d(P_1F_2)|$
- $|d(P_2F_1)-d(P_2F_2)|$
- $|d(P_3F_1)-d(P_3F_2)|$

c. What is your conclusion?

d. Write a rule that will define all the points on the curve. (**Note:** The fixed points F_1 and F_2 are the **focal points** of the curve.)

e. Give the locus definition for the set of points on this curve.

f. What effect will it have on the curve if the distance between the foci is decreased and the constant of the difference is fixed? What restriction applies?

Check your answers by turning to the Appendix.

In the next example you will draw the curve of a quadratic relation from its locus definition.

Example 2

On a double-concentric-circle graph paper, draw a curve so that the sum of the distances each point of the curve lies from two fixed points is 16.

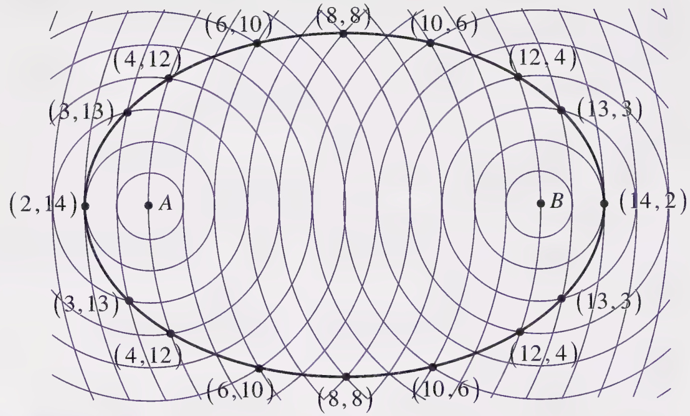
Solution

Step 1: Label the two centres A and B.

Step 2: Find a point P such that the sum of $d(PA)$ and $d(PB)$ is equal to the selected number. If you use the point $(10, 6)$, you see that $10 + 6 = 16$. To locate a point, find the intersection of the concentric circles from A and B. The point $(10, 6)$ is the intersection of the tenth concentric circle from A and the sixth concentric circle from B. You will have a point on either side of AB.

Step 3: Find and label other points that satisfy this condition. Some of these points are $(2, 14)$, $(3, 13)$, $(4, 12)$, $(6, 10)$, $(8, 8)$, $(12, 4)$, $(13, 3)$, and $(14, 2)$.

Step 4: Connect the points and you will get a curve. The resulting curve is an ellipse.



This curve satisfies the locus definition for an ellipse. That is, the sum of the distances from any point on the curve to two fixed points is a constant.

5. **a.** Use the double-concentric-circle graph paper at the back of the Appendix to draw the following curve.

Step 1: Label the two centres A and B.

Step 2: Select a number between 1 and 20.

Step 3: Find a point P such that $|d(PA) - d(PB)|$ is equal to the number you selected in Step 2.

Step 4: Find and label other points that satisfy this condition.

Step 5: Continue this process until you can draw the complete curve.

b. What shape is this curve?

c. Why does this procedure produce this curve?

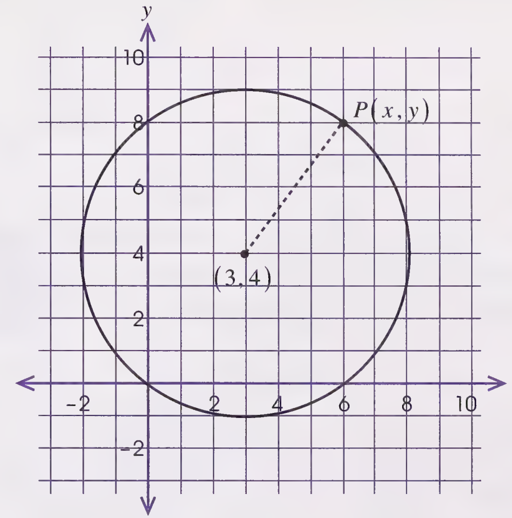
Check your answers by turning to the Appendix.

You will now be able to find the equation of each conic you studied from its locus definition and its graph. Study the following example.

Example 3

Determine the equation for the locus of all points which are 5 units from a fixed point $C(3, 4)$.

Solution



Let point $P(x, y)$ be any point on the circle. The distance formula is used to find the expression for the distance from point P to the centre of the circle.

This distance is the radius of the circle. The distance formula is as follows:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In this problem, d is the radius (5 units).

$$\sqrt{(x-3)^2 + (y-4)^2} = 5$$

If both sides are squared, the result is as follows:

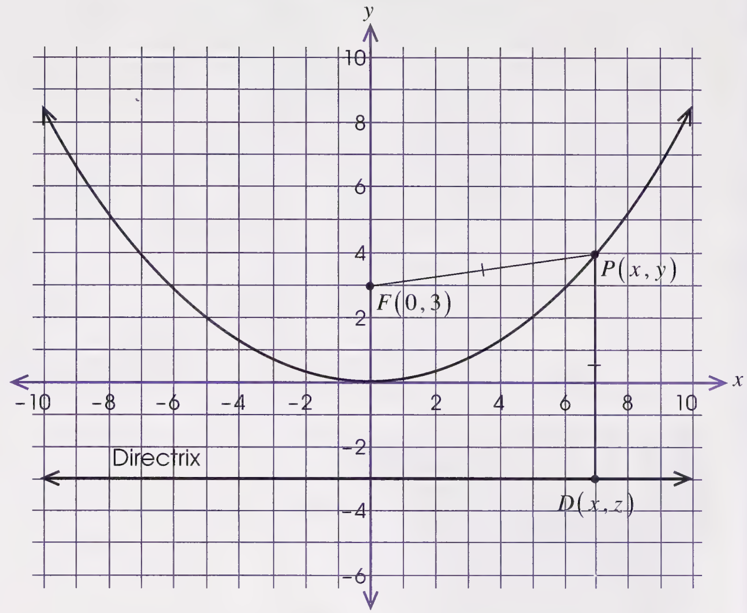
$$(x-3)^2+(y-4)^2=25$$
$$x^2-6x+9+y^2-8y+16=25$$
$$x^2+y^2-6x-8y=0$$

In Section 1 you discovered that in the quadratic equation for a circle centred at the origin, $A=C$ and $B=D=E=0$.

The circle shown in the graph with centre $(3,4)$ passes through the origin. Therefore, $F=0$.

The locus with centre $(3,4)$ and radius 5 cm is a circle defined by $x^2+y^2-6x-8y=0$, where $A=C=1$, $B=0$, $D=-6$, and $E=-8$.

6. $P(x,y)$ is any point on the following parabola. $D(x,z)$ is a point on the directrix. The coordinates of F are $(0,3)$.



a. What is the value of z?

b. Write the equation of the directrix.

c. Use the locus definition to determine the equation of this parabola.

7. Look at the sketch of the ellipse that was constructed in question 3 of Activity 1.

   Draw a set of axes so that the ellipse is centred at the origin. Label the coordinates of the two focal points F_1 and F_2. Note that for this curve $d(CF_1)+d(CF_2)=10$. Determine the equation of this ellipse.

8. Look at the sketch of the curve that was drawn in question 4 of Activity 1.

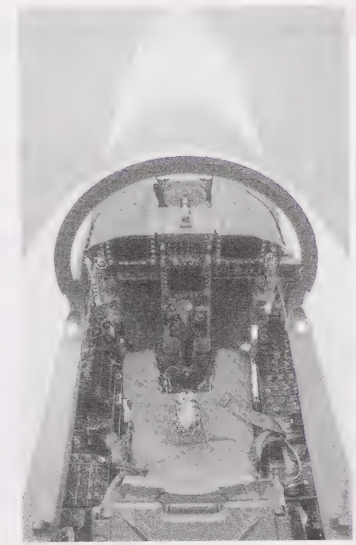
   Label the coordinates of the foci, and write the equation of this hyperbola if $\left|d(PF_1)-d(PF_2)\right|=4$.

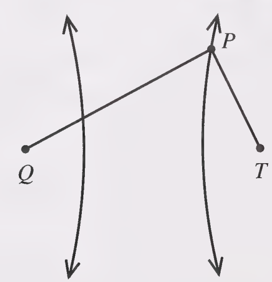
You can verify the locus definition of a curve if its equation and other related information is known. You will discover this in the following question.

9. The equation of a curve with directrix $y=-4$ and focus $(0,2)$ is $x^2-12y-12=0$. By selecting two points on the curve, verify its locus definition.

Check your answers by turning to the Appendix.

A useful application of the hyperbola is in the navigation of ships and planes. The (*lo*ng *ra*nge *n*avigation) loran system of navigation is used for determining the locations of ships and planes. Radio signals are sent by two sending stations Q and T which are located as shown in the following diagram. Point P represents the location of a plane on one branch of the hyperbola.



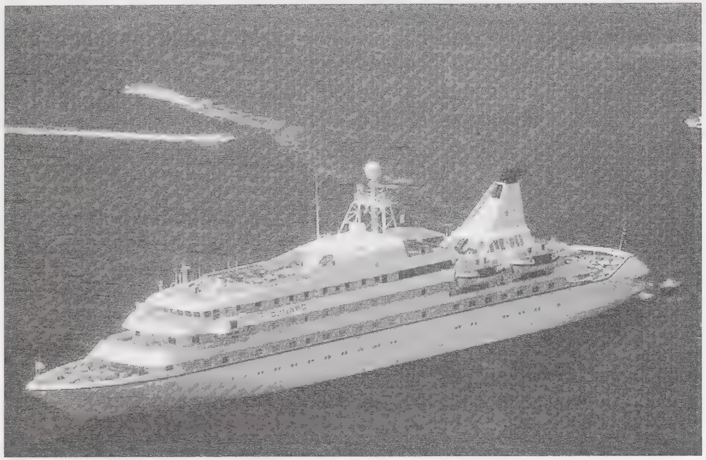


The time difference in the signals is changed to a distance difference $\left|d(PQ)-d(PT)\right|$ by computers on the plane. This distance difference is always a constant value. The focal points are Q and T.

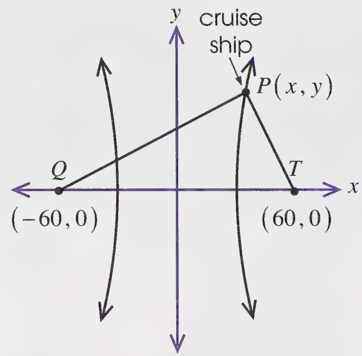
The following example illustrates the loran system.

Example 4

Two loran sending stations are 120 km apart. A cruise ship is 40 km closer to Station T than it is to Station Q. Determine the equation of the hyperbolic path on which the ship is located.



Solution



The sending stations can be located on either the x- or y-axis. In this example the x-axis will be used. The coordinates of the sending stations are $(\pm 60, 0)$ since the stations are 120 km apart. The constant difference is 40 km. In other words, $|d(PQ) - d(PT)| = 40$. According to the distance formula and the locus definition for a hyperbola, you can determine the equation of the hyperbolic path.

$$|d(PQ) - d(PT)| = 40$$

$$\sqrt{[x-(-60)]^2 + (y-0)^2} - \sqrt{(x-60)^2 + (y-0)^2} = \pm 40$$

$$\sqrt{(x+60)^2 + y^2} = \pm 40 + \sqrt{(x-60)^2 + y^2} \quad \text{(Square both sides.)}$$

$$x^2 + 120x + 3600 + y^2 = 1600 \pm 80\sqrt{(x-60)^2 + y^2} + x^2 - 120x + 3600 + y^2$$

$$240x - 1600 = \pm 80\sqrt{(x-60)^2 + y^2} \quad \text{(Divide each term by 80.)}$$

$$3x - 20 = \pm\sqrt{(x-60)^2 + y^2} \quad \text{(Square both sides.)}$$

$$9x^2 - 120x + 400 = x^2 - 120x + 3600 + y^2$$

$$8x^2 - y^2 - 3200 = 0$$

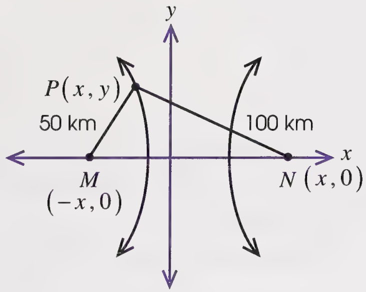
The equation of the hyperbola is $8x^2 - y^2 - 3200 = 0$.

Use your knowledge to answer the following questions.

10. Describe the locus for each of the following.

a. a seat on a revolving Ferris wheel
b. a spider at the end of a swinging pendulum
c. the hub of a wheel on a flat surface

11. A plane is 100 km from Station M and 50 km from Station N as shown in the diagram.



How far apart are the two stations?

12. $F_1(-4, 0)$ and $F_2(4, 0)$ are two fixed points. If all the points on the locus satisfy $|d(PF_1) - d(PF_2)| = 6$, show that $(9, 6)$ does not lie on the curve.

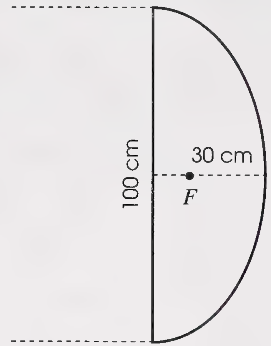
13. A physical education teacher lays out an elliptical design in the sports field using a cord and two pegs. The ellipse is 6 m long and 4 m wide.

a. What length of cord is required?

b. How far from the centre should each peg be? Give your answer as an exact value.

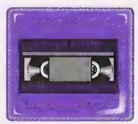
14. An ellipse with foci $(-4, 0)$ and $(4, 0)$ is defined by the equation $9x^2 + 25y^2 - 225 = 0$. By selecting two points on the locus, verify the locus definition for this curve.

15. A searchlight reflector is designed so that a cross section through its axis is a parabola and the light source is at the focus. The diameter of the reflector is 100 cm and the reflector is 30 cm deep. Find the distance from the vertex to the light source.



Check your answers by turning to the Appendix.

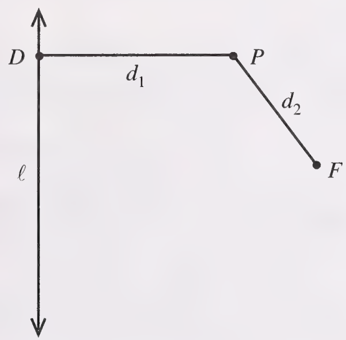
Curves are traced by a moving point under a given condition, such as the bicycle wheel tracing a path parallel to the hub of the wheel. In this activity you described each conic section in terms of its locus.



View the video titled *Defining The Locus* from the *Discovering Conics* series, ACCESS Network. This video gives the locus definition and shows how to graph a parabola, an ellipse, and a hyperbola using double-concentric-circle graph paper, and the locus definitions. This video is available from the Learning Resources Distributing Centre.

Activity 2: Using Eccentricity to Define Conics

If ℓ is a fixed line (**directrix**), F is a fixed point (**focus**) not on that line, and P is any point, it is possible to measure the distance from P to ℓ (**distance** ***PD***) and from P to F (**distance** ***PF***). If the ratio of the distance between P and F to the distance between P and ℓ is a constant, then the locus of this point P is a conic.



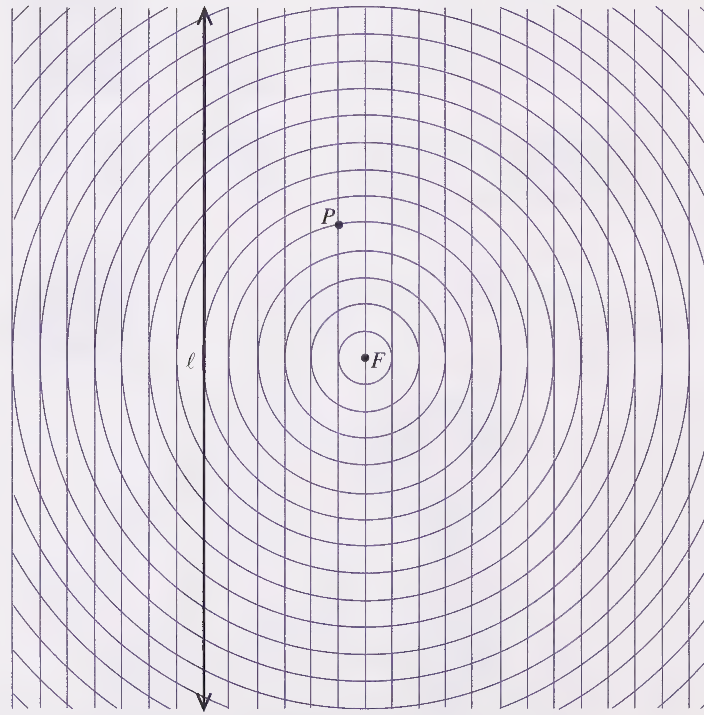
There is not only a special name for the fixed point and the fixed line, but also the ratio. The ratio of the two distances ($\frac{d_2}{d_1}$, as shown in the preceding diagram) is called the **eccentricity** (symbol $\boldsymbol{e}$).

$$e = \frac{d_2}{d_1} \quad \text{or} \quad e = \frac{d(PF)}{d(PD)}$$

How does eccentricity determine the shape of conics?

To find the answer look at the following example.

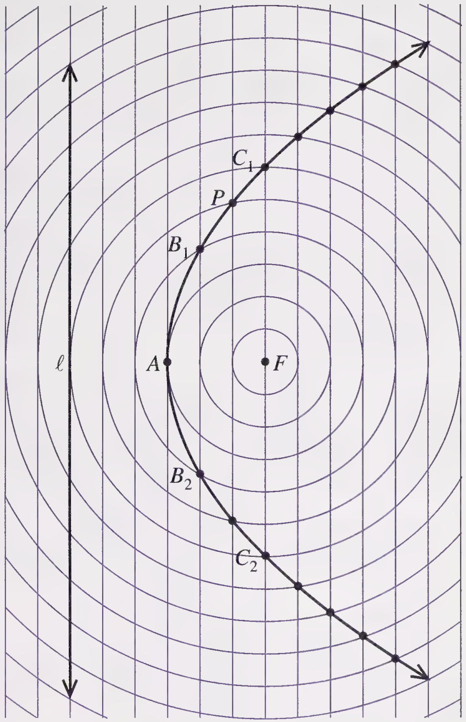
Example



A sheet of circle-line graph paper shows the focus F at the centre of the concentric circles. The point P is the intersection of a circle and a line. The line ℓ is the directrix 6 units from the centre.

On the plane, find as many points as possible that determine a conic with an eccentricity of 1. What conic section do these points represent?

Solution



P is 5 units from F and ℓ. The radii of the concentric circles are increased by the same unit as the vertical lines. Therefore, if the number of circles from the centre is equal to the number of lines from the directrix, then the point of intersection of the last circle and the last vertical line is the point equidistant from the focus and directrix. Locate point A, 3 units from F and ℓ; point B_1, 4 units from F and ℓ; B_2, 4 units from F and ℓ; C_1, 6 units from F and ℓ; C_2, 6 units from F and ℓ, and so on. Join the points. The points determine a parabola.

View the video titled *Eccentricity* from the *Discovering Conics* series, ACCESS Network. This video shows how to draw a parabola, an ellipse, or a hyperbola using concentric-circle graph paper and using a given eccentricity value. This video is available from the Learning Resources Distributing Centre.

1. Take a sheet of circle-line graph paper from the Appendix. Let the centre of the concentric circles represent a fixed point F (focus), and choose any one vertical line (perhaps a line 5 units from F) to represent the directrix. Find any point P such that the ratio of d_2 (the distance from P to F) to d_1 (the distance from P to the directrix) is $\frac{2}{3}$. (**Hint:** $\frac{2}{3}$ is the same as $\frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots, \frac{d_2}{d_1}$. Thus, $d_2 = 4$ when $d_1 = 6$, $d_2 = 6$ when $d_1 = 9$, and so on.) What conic do these points determine?

2. Repeat question 1 using different ratios (eccentricity) between 0 and 1. (**Hint:** You may find it difficult to sketch the graph if $e < \frac{1}{2}$.) Compare the shapes of the ellipses. What other conjectures can be made?

Check your answers by turning to the Appendix.

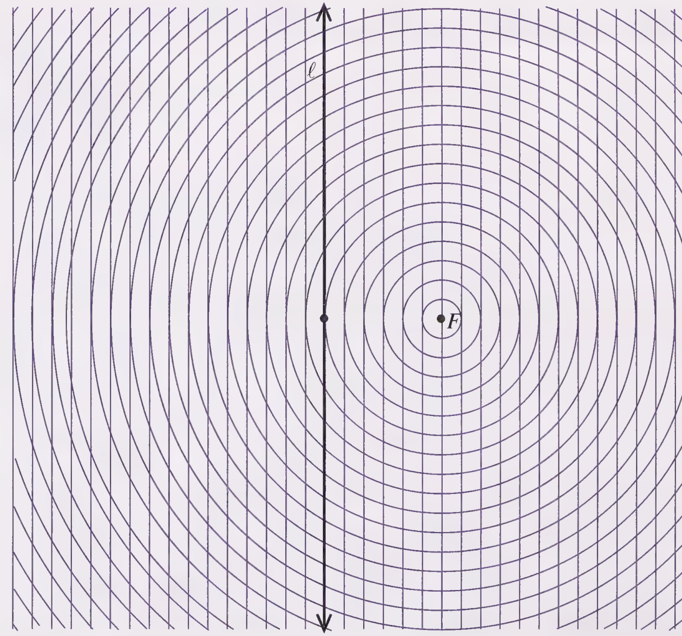
If you finished question 2, you should be able to tell that the eccentricity measures the shape of an ellipse. As e approaches 1, the ellipse approaches a line segment. On the other hand, as e approaches 0, the ellipse approaches a circle. In other words, the limiting position of an ellipse is a circle as e approaches 0, and the limiting position of an ellipse is a line segment as e approaches 1. The length of this segment would be twice the length of the radius of the circle. Since the limiting position of an ellipse is a circle as e approaches 0, you can say that a circle is a special ellipse.

3. Take a sheet of circle-line graph paper from the back of the Appendix. Let the centre of the concentric circles represent the focus. Let $e = \frac{3}{4}$. Use three different vertical lines to represent three different directrices. Sketch the three different ellipses on the same sheet of graph paper. What happens to the ellipses? Summarize your findings in a paragraph.

Check your answers by turning to the Appendix.

Example 2

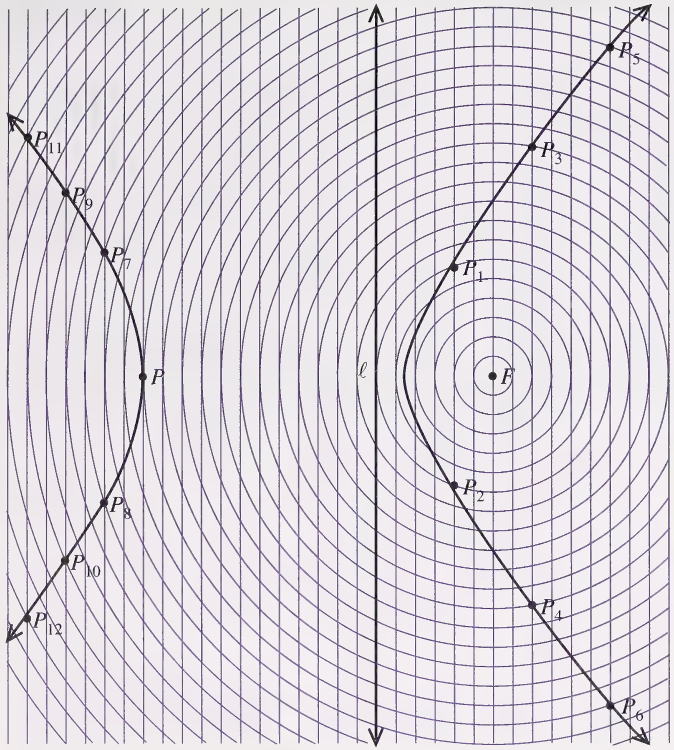
On the following circle-line graph paper, the centre of the concentric circles represents the focus F. A vertical line which is 6 units from the focus represents the directrix ℓ. Find any point P such that the ratio of $d(PF)$ to $d(P\ell)$ is $\frac{3}{2}$. What conic do these points determine?



Solution

$\frac{3}{2}$ is the same as $\frac{6}{4}, \frac{9}{6}, \frac{12}{8}, \frac{15}{10}, \frac{18}{12}, \frac{21}{14}, \ldots, \frac{d(PF)}{d(P\ell)}$. The point P_1 is 6 units from F and 4 units from ℓ. P_2 is also 6 units from F and 4 units from ℓ. P_3 and P_4 are 12 units from F and 8 units from ℓ. P_5 and P_6 are 18 units from F and 12 units from ℓ. Join these points with a smooth curve.

It is also possible to find the points of intersection on both sides of the directrix. P is 18 units from F and 12 units from ℓ. P_7 and P_8 are 21 units from F and 14 units from ℓ. P_9 and P_{10} are 24 units from F and 16 units from ℓ. P_{11} and P_{12} are 27 units from F and 18 units from ℓ. Join these points with a smooth curve. The points determine a hyperbola. The eccentricity of a hyperbola is $e > 1$.



4. Repeat the question in Example 2 using the same focus and same directrix, but different eccentricities which are greater than 1. Sketch all conics on the same graph paper. What is your conclusion?

5. Repeat the question in Example 2 using the same focus and the same eccentricity, but different directrices. Sketch all conics on the same graph paper. What conjectures can be made?

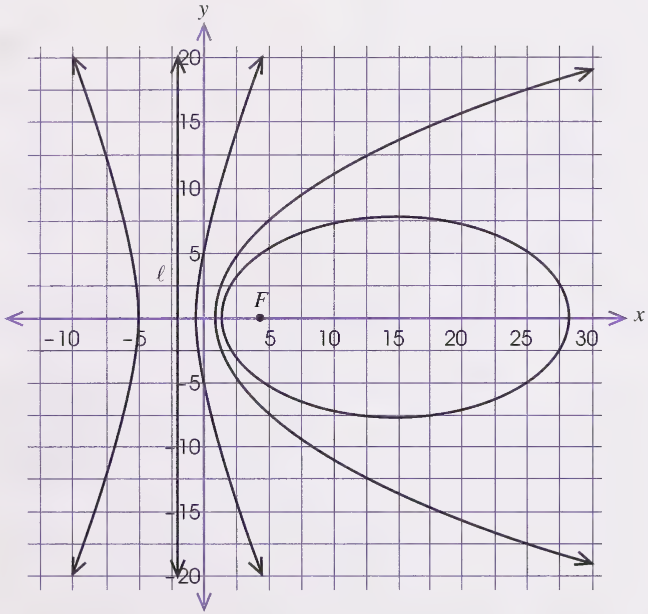
If you put the focus and directrix back to an ordinary coordinate system, can the equation of the locus be found when eccentricity is given? The answer is in the next investigation.

6. Take a sheet of ordinary graph paper from the back of the Appendix. Draw the x- and y-axes. Locate the point $F(2, 3)$ and draw the vertical line $x = -1$. If $P(x, y)$ is any point on a conic and the ratio of d_2 (the distance from P to F) to d_1 (the distance from P to the vertical line) is $\frac{3}{2}$, use the distance formula and the definition of eccentricity to find the equation of the locus. What conic does this equation represent?

Use a graphing calculator to answer the following question. You may use a computer program if you wish.

7. Repeat question 6. Use the same directrix and focus. First change the eccentricity to 1; then change it to $\frac{3}{4}$. Use the equations to graph the conics. Use eccentricity to define the three conics: the parabola, the ellipse, and the hyperbola.

8. The following graph shows a parabola, an ellipse, and a hyperbola. They have the same focus and directrix. Identify a point on each conic and use this point to determine the eccentricity of the conic. Check your answer by choosing another point and repeating the calculation.



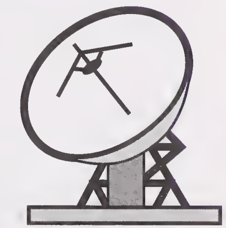
With the help of the previous questions, answer the following.

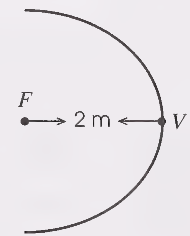
9. The eccentricity of a parabola is always equal to 1. Does this mean that the eccentricity does not affect the shape of a parabola? Identify and explain how the other factors affect the shape of a parabola.

10. How does the eccentricity affect the shape of an ellipse?

11. What is the effect of a smaller eccentricity on the shape of a hyperbola?

12. A large parabolic antenna is shown in the diagram. A cross section of the antenna is shown. Using the dimension shown on the diagram, find an equation of the reflector. (Place the vertex at the origin.)





13. a. The distance between a directrix and a focus is 4 units. Use a circle-line graph (from the Appendix) to sketch the conic which has an eccentricity of 3.

b. What conic is it?

14. The vertical line $x = \frac{16}{5}$ is a directrix. The point $P(5, 0)$ is a focus. The eccentricity of the conic is $\frac{5}{4}$. Prove that the point $(4, 0)$ is a point on the conic.

Check your answers by turning to the Appendix.

Eccentricity changes the size and shape of a conic section. In this activity you discovered that the equation of a conic can be determined from its eccentricity.

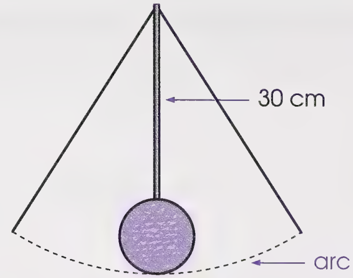
Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

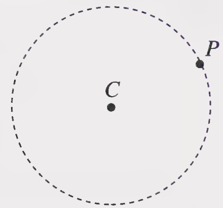
Extra Help

A locus of points is a set of points which, under a given condition, describes lines or curves.

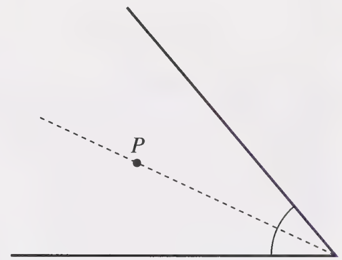
- The path described by the end of a swinging pendulum is an arc. The condition is the length of the pendulum.



- The point P moves such that it is equidistant from a fixed point C. The locus is a circle and the condition is the radius PC.



- The locus of point P equidistant from two lines is the bisector of the angle formed by the two lines.



1. Describe the locus of each of the following:

   a. taking an elevator
   b. a speck of dust on a revolving record
   c. sitting in the seat of a rotating Ferris wheel
   d. two children playing on a seesaw

2. Sketch and state the locus of each of the following:

   a. all points 8 cm from a line

   b. the path of a lead ball tied to the end of a rope and swung by a person

   c. all points equidistant from two intersecting lines

Check your answers by turning to the Appendix.

Enrichment

In your study of the conic sections you gained familiarity with the general equation of the second degree, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. A conic can be identified by examining the coefficients in the second degree equation.

The following theorems will help you to classify the four primary conic sections and the degenerate conic sections.

- If $\boldsymbol{B^2 - 4AC < 0}$, the curve is an ellipse, a circle, a point, or there is no curve.
- If $\boldsymbol{B^2 - 4AC > 0}$, the curve is a hyperbola or two intersecting straight lines.
- If $\boldsymbol{B^2 - 4AC = 0}$, the curve is a parabola, two parallel lines, one line, or there is no curve.

Example 1

What conic is defined by the equation $x^2 - 4y^2 - 8 = 0$?

Solution

Substitute the values of the variables in the equation.

$$\begin{aligned} B^2 - 4AC &= (0)^2 - 4(1)(-4) \\ &= 16 \end{aligned}$$

Since $B^2 - 4AC > 0$, the equation may represent a hyperbola or two intersecting lines.

1. What conic is represented by each equation? Explain.

   a. $x^2 + y^2 - 8y + 7 = 0$

   b. $5y^2 + 8x - 2y - 6 = 0$

Check your answers by turning to the Appendix.

You can use the formula $B^2 - 4AC$ further to narrow it down to some specific curves, as follows:

- If $B^2 - 4AC < 0$ and $A = C$, the curve is a circle.
- If $B^2 - 4AC < 0$ where $A \neq C$, and A and C are both positive, the curve is an ellipse.
- If $B^2 - 4AC > 0$ where A and C have opposite signs, the curve is a hyperbola.

Example 2

What conic is represented by the following equation?

$$2x^2 + y^2 - 8x - 8y + 16 = 0$$

Solution

Substitute the values of the variables in the equation and examine the values of A and C.

$$\begin{aligned} B^2 - 4AC &= (0)^2 - 4(2)(1) \\ &= -8 \end{aligned}$$

Since $B^2 - 4AC < 0$, $A \neq C$, and A and C have the same sign, the curve is an ellipse.

2. What conic is represented by each equation? Explain.

   a. $2x^2 + 2y^2 - 12x + 20y + 36 = 0$

   b. $4y^2 - 9x^2 - 36 = 0$

   c. $3x^2 + 5y^2 - 15 = 0$

Check your answers by turning to the Appendix.

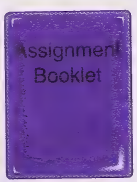
Conclusion

In this section, you recognized that each conic can be described as a locus of points. You used these locus definitions to derive equations of conics, and you solved related real-world problems.

As well as using the locus definitions, you looked at how conics can be described using eccentricity—the ratio of the distances from the curve to a fixed point and a fixed line.

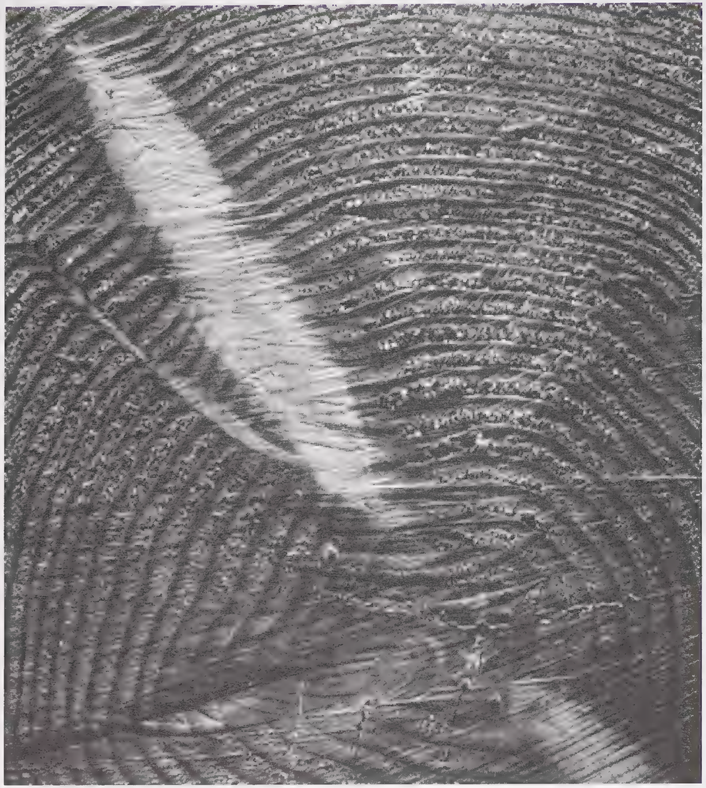
A goat that grazes in a circle because it is tethered to a stake is a mundane example of a real-world application of the locus definition of the circle. In fact, Leonardo da Vinci used an arc of a circle to paint Mona Lisa's enigmatic smile; the arc connects four key points—the ends of her lips and the outer corners of her eyes. You can also find patterns of conics in such things as paintings, jewellery, and of course, all technology that enriches your life.

Assignment



You are now ready to complete the section assignment.

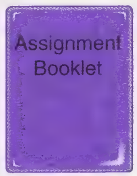
Module Summary



In this module, you studied quadratic relations from four different points of view. Conics were generated by a double-napped cone cut by a plane in various ways. You investigated the changes in the coefficients of the general quadratic equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $B = 0$, and how these changes affected the conics. The locus definition was used to define the conics and derive their equations. You also had to describe conics using eccentricity.

Geometric patterns such as the path of an object thrown into the air, concentric circles of tree trunks, and the path of a boomerang are some of the phenomena that can be described using quadratic relations. Because of this, you are able to see the connection between real-world applications and their algebraic representations. How many examples of conic sections can you see in the preceding photograph?

Final Module Assignment



You are now ready to complete the final module assignment.

APPENDIX

- Glossary
- TI-81 Graphing Calculator
- Suggested Answers
- Graph Paper

Glossary

Circle: a closed plane curve where every point is equidistant from a fixed point within the curve

Conic: any curve which is the locus of a point that moves so that the ratio of its distance from a fixed point to its distance from a fixed line is constant

Conic sections: shapes that occurs when a double-napped cone is intersected by a plane

Degenerate conic: a point, a straight line, or a pair of straight lines, which is a limiting form of a conic

Eccentricity: the ratio of the distance between a point on the curve and the focus to the distance between that same point on the curve and the directrix

Ellipse: a closed plane curve generated by a point moving in such a way that the sum of its distances from two fixed points is a constant

Generator: a rotating line which sweeps out a surface according to some law

Hyperbola: the set of points whose distances from two fixed points, the foci, have a constant difference

Nappe: one of the two parts of a conical surface into which the surface is separated by the vertex

Parabola: a plane curve which is the set of all points equidistant from a fixed point and a fixed line in the plane

Parameters: arbitrary constants or variables in a mathematical expression that distinguishes various specific cases

TI-81 Graphing Calculator

This section provides a sample program for the TI-81 graphing calculator. Carefully read all of this sample program and the key strokes chart before you begin programming your calculator; then return to this point and start entering the program. If at some point while entering the program you leave the calculator on long enough that it automatically shuts off, just press ON and your program will return.

To enter the program turn the calculator on, push PRGM, then press ▶ to EDIT. You should now be at Program 1. Press ▼ if you want to select a different program number. Press ENTER. Enter the word CONICS here. To enter the word CONICS, press each of the letters C O N I C S and ENTER. (**Note:** In the first program line, you do not need ALPHA to enter letters. After this first line you will need to press ALPHA before each letter you want to enter.)

The standard range setting of $x = -10$ to 10 and $y = -10$ to 10 for a graphing calculator does not give a perfect circle when a circle is graphed since the screen is wider than it is high. To get equal units on each axis, you need to set particular range values. These range values are given in the sample program for the TI-81.

Since you are using a program in the calculator, it will be best if the range values are set in the program. This way, even if you change the range outside the program, each time you go back to the program the range will reset to the same values.

Enter the range values, starting at the first blank line after you have entered the word CONICS in the PRGM # line. Enter −14.1, which is the minimum value you want for x. Be sure to use the (−) for negative. Next, press the STO▶ key, which is found in the lower left corner of the keyboard. Next, press the VARS key. This takes you to a new menu. In the right top corner on this new menu, you will find the abbreviation RNG. This stands for range. Press ◀ once. This will move the highlight on to the RNG abbreviation and give you a new menu displaying Xmin, Xmax, and so on.

Press 1 on the keyboard since you want to store −14.1 for Xmin. Then press ENTER to complete the instruction. Next, enter the value 14.1. Again press STO▶, then VARS, and then ◀ to highlight RNG. Now press 2 on your keyboard to store the value 14.1 as Xmax. Again, press ENTER to complete this line of instruction. Continue this sequence until you have entered all seven values into the program. When you have finished, your screen should have the first seven lines of your program on it. Now you are ready to enter the next line of the program.

To enter Disp "A", press PRGM, ▶ to I/O, and use ▲ or ▼ to select Disp. Since Disp is the first line of the menu, you may not have to move up or down. Press ENTER; then press ALPHA + for the quotation marks; then ALPHA and A for the letter A; then ALPHA + for one more set of quotation marks. Press ENTER to move to the next line. Be sure to press the keys firmly.

Remember: When entering the equations, be sure to use the (−) key for the negative sign and the − key for minus or subtraction.

Now enter the remainder of the program using the keystrokes listed in the keystrokes chart. Notice that most of the symbols require more than one keystroke. You may edit the program by using the arrow keys to move up or down or left or right. Use DEL to delete an entry. Use INS to insert a missing entry.

The following statements show exactly what should appear on your calculator screen as you enter the program.

```
Prgm 1:  CONICS
–14.1→Xmin
14.1→Xmax
1→X scl
–9.3→Ymin
9.3→Ymax
1→Y scl
1→res
:Disp "A"
:Input A
:Disp "B"
:Input B
:Disp "C"
:Input C
:Disp "D"
:Input D
:Disp "E"
:Input E
:Disp "F"
:Input F
:If C≠0
:Goto 3
:"(–1*A*X²–D*X–F)/(B*X+E)"→Y₁
:DispGraph
:End
:Lbl 3
:"(–1*(B*X+E)+√((B*X+E)²–4*C*(A*X²+D*X+F)))/(2*C)"→Y₁
:"(–1*(B*X+E)–√((B*X+E)²–4*C*(A*X²+D*X+F)))/(2*C)"→Y₂
:DispGraph
:End
```

Symbol	Keystrokes for TI-81
"	ALPHA +
A	ALPHA MATH
B	ALPHA MATRX
Disp	PRGM, ▶ to I/O, ▲ or ▼ to highlight Disp, ENTER
Input	PRGM, ▶ to I/O, ▲ or ▼ to highlight Input, ENTER
≠	2nd, MATH, arrow to ≠, ENTER
Y_1	2nd, VARS, arrow to Y_1, ENTER
*	×
exponent 2	x^2
$\sqrt{\ }$	2nd x^2

Symbol	Keystrokes for TI-81
/	÷
→	STO▶
If	PRGM, arrow to If, ENTER
Goto	PRGM, arrow to Goto, ENTER
DispGraph	PRGM, ▶ to I/O, ▲ or ▼ to highlight DispGraph, ENTER
End	PRGM, arrow to End, ENTER
Lb1	PRGM ENTER

When you have entered the program, push 2nd CLEAR. To run the program push PRGM, ▼ to choose the correct program, then ENTER, and then ENTER again. Now enter your values of *A*, *B*, *C*, *D*, *E*, and *F*. Press a number for *A* (for negative numbers use the (–) key) and ENTER; press a number for *B* ENTER; and so on until you have entered all six values.

Suggested Answers

Section 1: Activity 1

1. a. When the flashlight is held perpendicular to the wall, a circle appears.

   b. When it is held at an oblique angle, an ellipse, a parabola, or a hyperbola appears.

   c. When the flashlight is moved closer to the wall, the light curve decreases in size.

   d. When the flashlight is moved away from the wall, the light increases in size.

2. a. When the solution is held horizontally, the shape in the cup is circular.

   b. When the cup is tilted at an angle, the shape in the cup is elliptical.

3. a. When the cylinder is sliced perpendicular to the vertical axis, two parallel lines appear.

   b. When the cylinder is sliced at an angle to the axis, an ellipse appears.

4. Answers may vary.

a. You can produce a circle in the following ways:

- pouring liquid into an erect cone or cylinder
- mapping or printing using a circular object
- cutting a cone at right angles to its axis

b. You can produce a parabola in the following ways:

- cutting a cone at an angle so that the cut is parallel to a generator

- squirting water upwards with a garden hose and allowing the water to fall

c. You can produce an ellipse in the following ways:

- cutting a cylindrical pipe at an angle
- pouring a liquid substance into a tire standing on edge
- cutting a cone at an angle $\neq 90°$ parallel to a generator
- looping a string around two fixed pins and mapping a path

d. You can produce a hyperbola in the following ways:

- cutting a right double cone vertically with a blade
- drawing a special curve and reflecting it in a mirror

5. a. The water surface, when a litre of water is poured into an upright car tire, forms an ellipse.

b. The cross section of a pipe, when the pipe is cut at an angle of 45°, forms an ellipse.

c. If the pipe is cut at an angle less than 45°, the ellipse will have a greater degree of curvature. If the angle is greater than 45°, the ellipse will be more elongated.

6. When you cut off a slice parallel to the generator or side of the cup, you get a parabolic shape.

7. a. It is often said that a circle is the limiting case of an ellipse. The reason for this is that when you cut a cone at an angle, the cross section will depict an ellipse. If you make the angle of the cut decrease towards the horizontal, the ellipse will get more and more rounded. Eventually, when the angle is zero, you will have reached the limiting position of the ellipse. In other words, the shape changes from an ellipse to a circle. The point at which the ellipse ceases to be an ellipse and becomes a circle is called the limiting position of the ellipse.

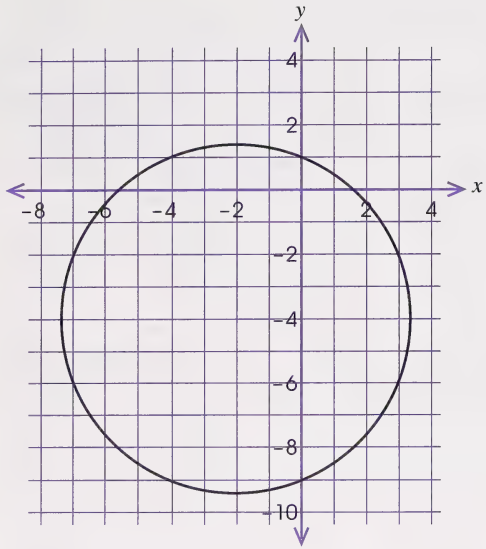
b. If a circle were drawn on a thin rubbery material, its shape would be changed when a force is applied pulling the rubbery sheet apart. The sheet would stretch and the circle would become an ellipse. Of course, the greater the pull, the more the elongation of the ellipse. You have to assume that the pull is uniform and evenly distributed.

8. In order to produce the conic sections, two right circular cones of equal size are needed. Then connect the vertices in such a way that the bases of the cones are pointing in opposite directions.

Use a plane to make various cuts vertically, horizontally, and obliquely.

Section 1: Activity 2

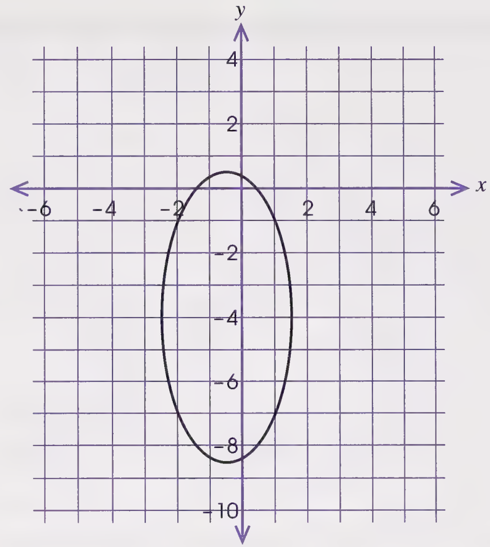
1. a.



b. The graph is a circle with a centre in the third quadrant.

c. The coefficients are $A=1$, $B=0$, $C=1$, $D=4$, $E=8$, and $F=-10$.

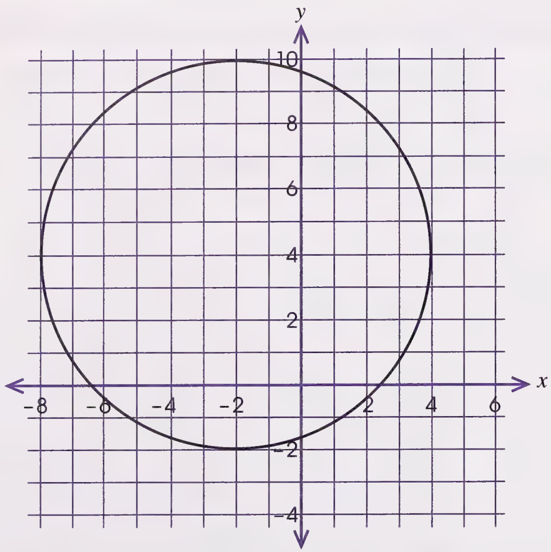
2. a.



b. The graph is an ellipse with its centre in the third quadrant.

c. The coefficients are $A=4$, $B=0$, $C=1$, $D=4$, $E=8$, and $F=-3$.

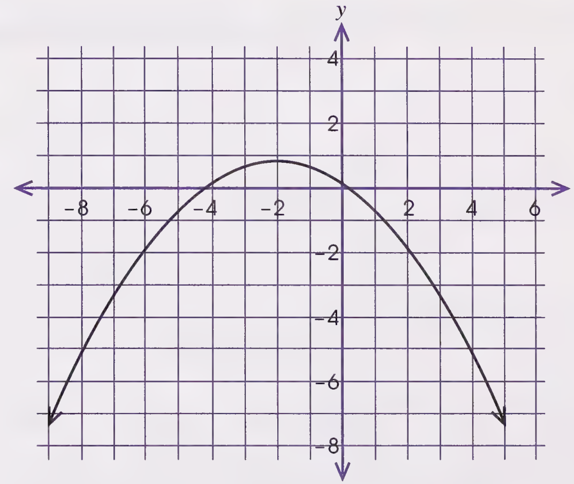
3. a.



b. The graph is a circle with its centre in the second quadrant.

c. The coefficients are $A=1$, $B=0$, $C=1$, $D=4$, $E=-8$, and $F=-16$.

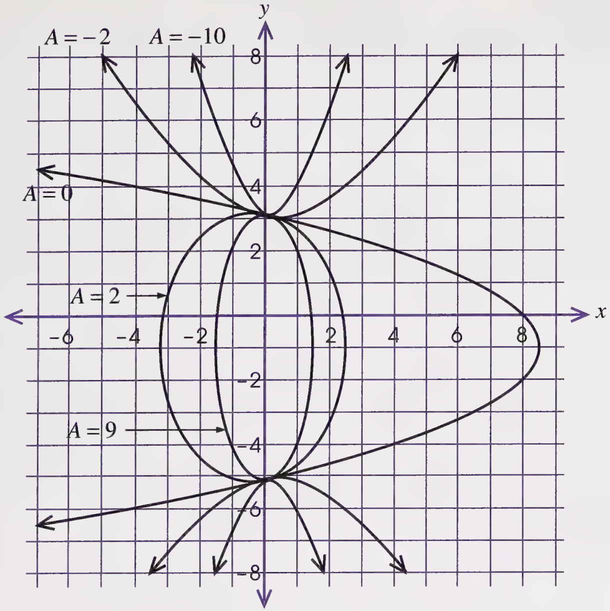
4. a.



b. The graph is a parabola with its vertex in the second quadrant.

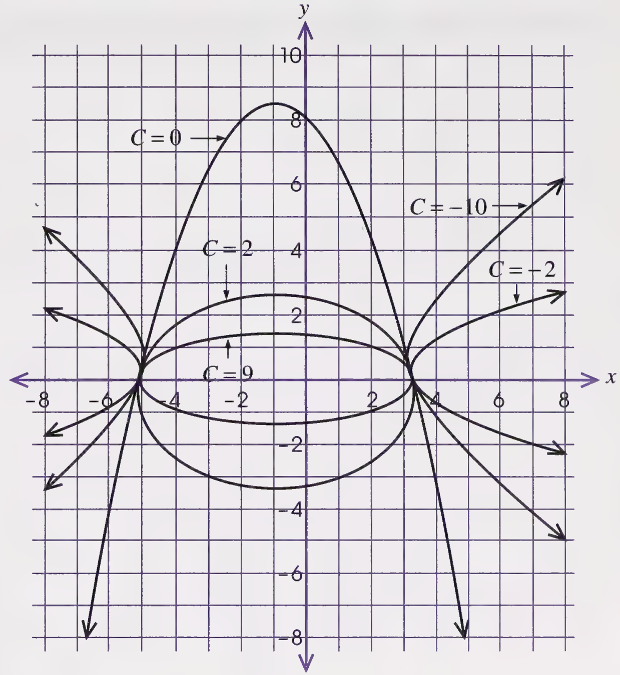
c. The coefficients are $A=1$, $B=0$, $C=0$, $D=4$, $E=6$, and $F=-1$.

5. a.



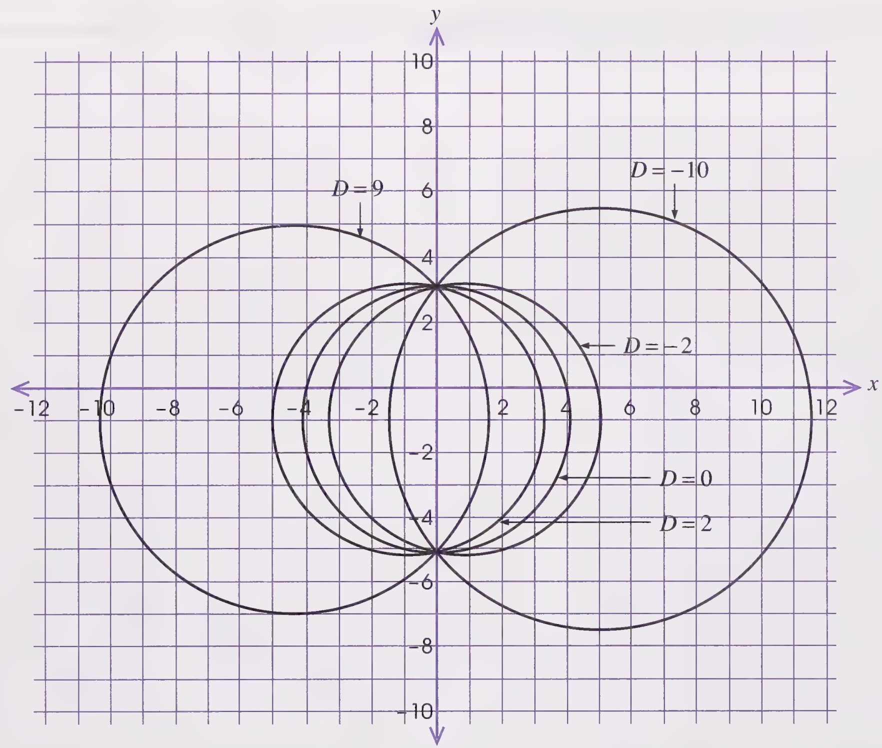
As A changes from -10 to 9, the curve changes from hyperbolas (when A is negative) to a parabola (when A is zero) to ellipses (when A is positive), and to a circle when $A=1$. Therefore, changing the value of A may change the type of curve as well as the shape of the curve.

b.



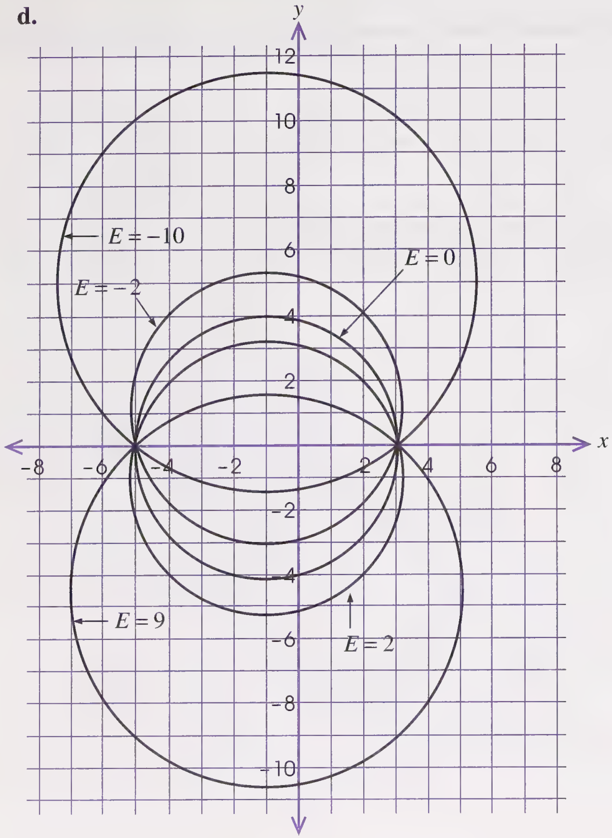
As C changes from -10 to 9, the curve changes from hyperbolas to ellipses. The changes are similar to those that occurred when A was changed. When $C=1$, the curve is circular. Therefore, changing the value of C may change the type of curve as well as the shape of the curve.

c.



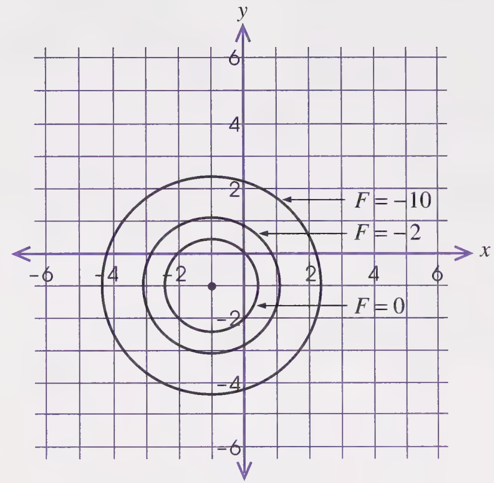
As D changes from -10 to 9, the curve moves (left) from having its centre in the fourth quadrant to having its centre in the third quadrant. The type of curve does not change; however, the size of the curve is changing. The curve gets smaller as the absolute value of D decreases. Therefore, changing the value of D appears to affect only the size and position of the curve, not the type or shape.

d.



As E changes from -10 to 9, the curve moves (downward) from having its centre in the second quadrant to having its centre in the third quadrant. The changes are similar to the changes for D. Therefore, changing the value of E appears to affect only the size and position of the curve, not the type or shape.

e.

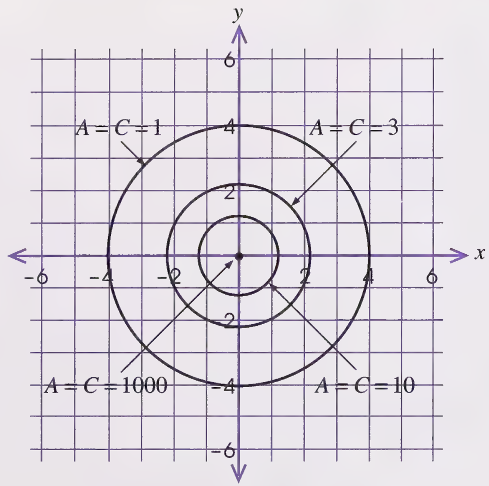


As F changes from -10 to 0, the curve gets smaller. When F is greater than zero, no graph is possible. Therefore, changing the value of F appears to affect the size of the circle. Some values of F will result in the degenerate case.

Section 1: Activity 3

The solutions in this activity contain possible values for the parameters in question. You may have different values, but you should have the same answers to the questions, and you should have similar conclusions.

1.

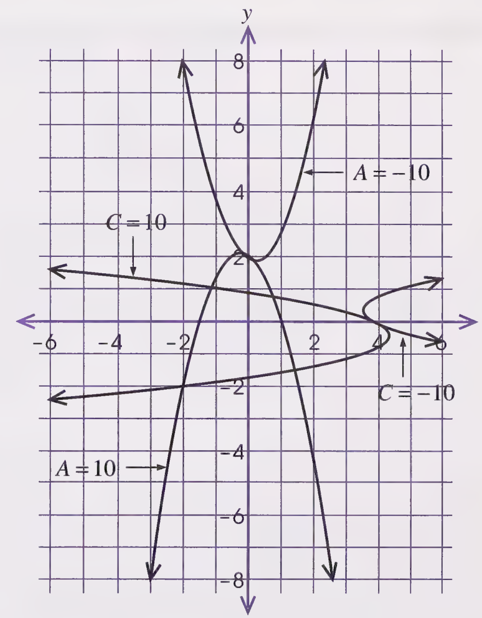


a. The resulting figure is a circle.

b. The circle decreases and eventually disappears when A and C are very large. Increasing values of A and C produce a smaller circle.

c. No circle is produced when A and C are negative.

2.



a. The resulting shape when either $A = 0$ or $C = 0$ is a parabola.

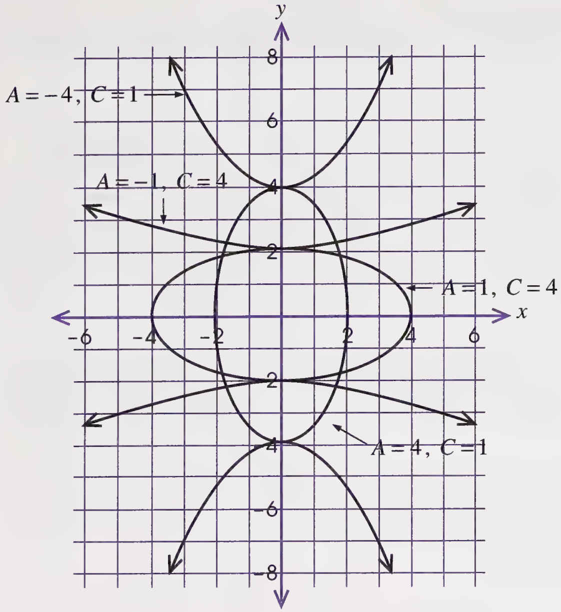
b. When $A > 0$ and $C = 0$, the parabola opens down.

c. When $A < 0$ and $C = 0$, the parabola opens up.

d. When $C > 0$ and $A = 0$, the parabola opens left.

e. When $C < 0$ and $A = 0$, the parabola opens right.

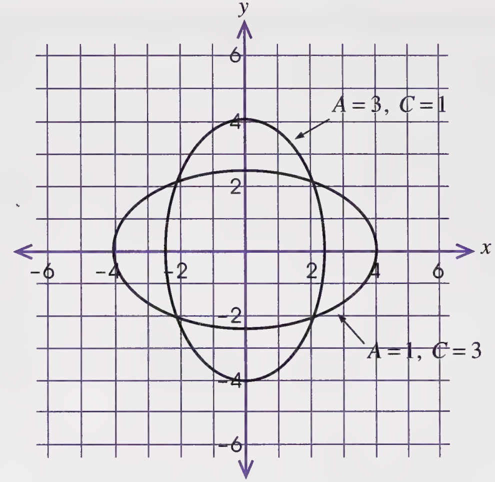
3.



a. The resulting shapes are ellipses when A and C are both positive. If A and C are both negative, no graph results.

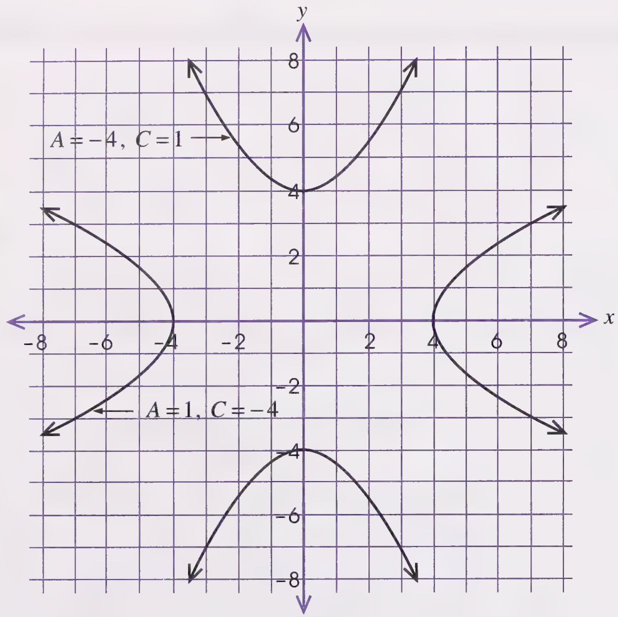
b. The resulting shapes are hyperbolas when A and C have different signs.

4.



When A is greater than C, the ellipse has its longer axis along the y-axis. When A is less than C, the ellipse has its longer axis along the x-axis.

5.

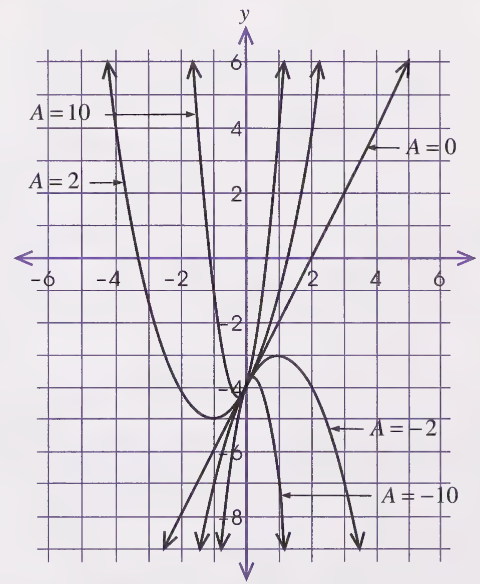


When A is positive and C is negative, the hyperbola opens to the left and right. When A is negative and C is positive, the hyperbola opens up and down.

6. When $A = C$ and F is constant, a circle results. Larger values of A and C result in smaller circles. When $A = 0$ or $C = 0$, the resulting curve is a parabola. When A and C are not equal but both are positive, an ellipse is the resulting curve. When A and C are not equal and one is positive while the other is negative, a hyperbola results.

Section 1: Activity 4

1.

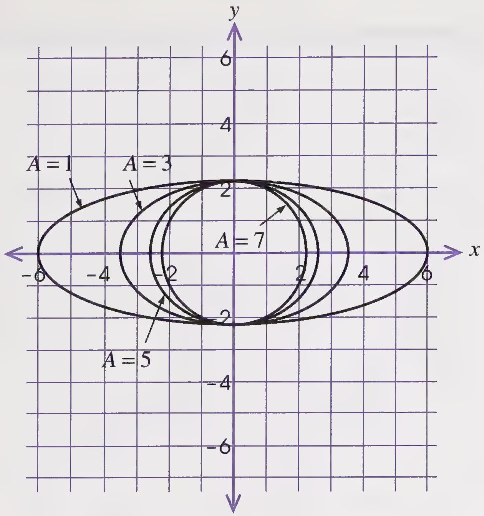


a. The graph opens down when $A < 0$ and up when $A > 0$. (When $A = 0$, a straight line is produced.)

b. The parabola is narrower when the absolute value of A is larger.

c. The vertex of the parabola shifts both horizontally and vertically as A changes.

2.



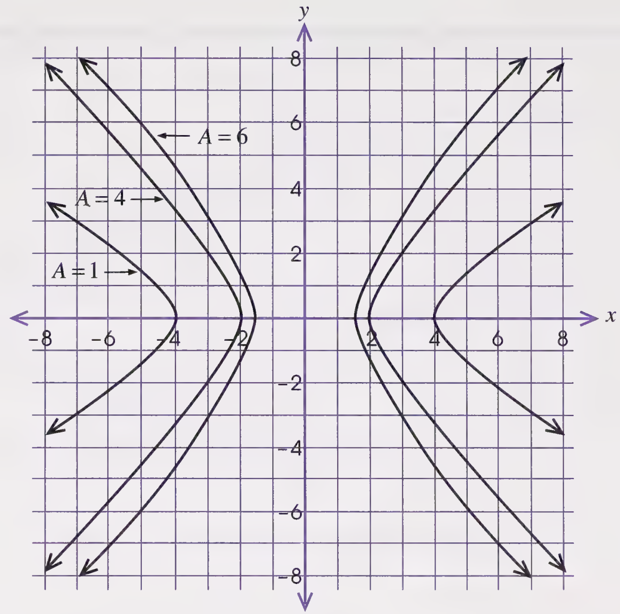
a. As the value of A increases from 1 to 3 to 5, the longer axis of the ellipse becomes shorter.

b. As A increases in size, the ellipse becomes more like a circle.

c. Letting $A = 7$ would result in the curve becoming a circle.

d. The ellipse will have its longer axis along the y-axis, and as the value of C increases from 1 to 3 to 5, the ellipse becomes more like a circle. At $C = 7$ the resulting curve will be a circle.

3.



a. As the value of A increases from 1 to 4 to 6, the hyperbola becomes wider and the two branches move closer together.

b. The hyperbola is on the y-axis, but the effects of increasing the size of C are the same.

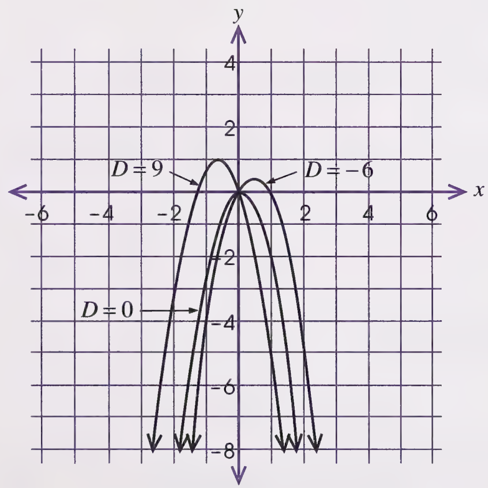
4. The parabola becomes narrower as the value of A or C increases.

The ellipse becomes more circular as the difference between A and C decreases. The ellipse becomes more elongated as the difference between A and C increases.

The arms of the hyperbola become flatter as the value of A or C increases.

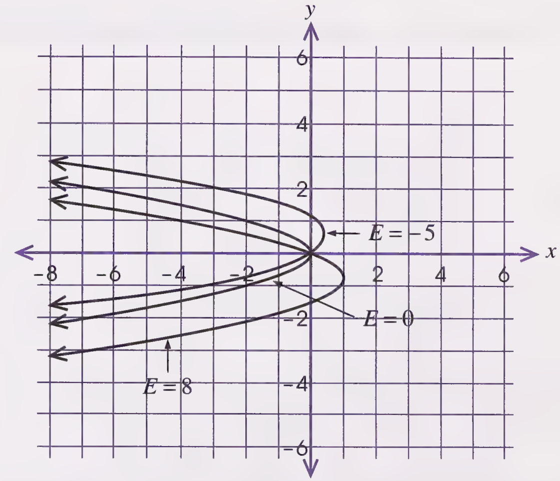
Section 1: Activity 5

1.



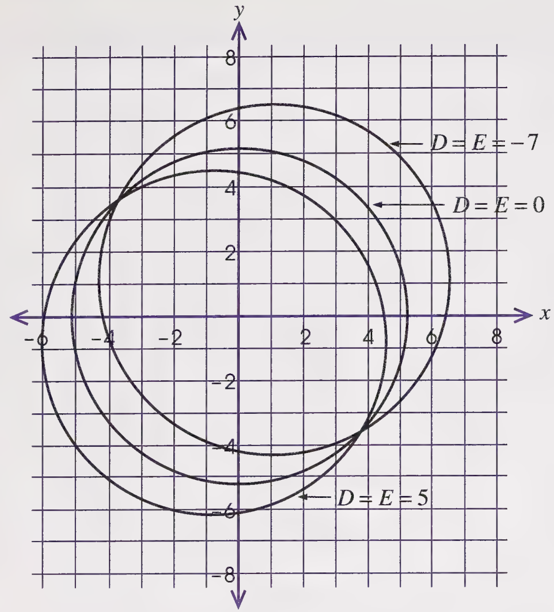
$D = 0$ puts the vertex of the parabola at the origin.

2.



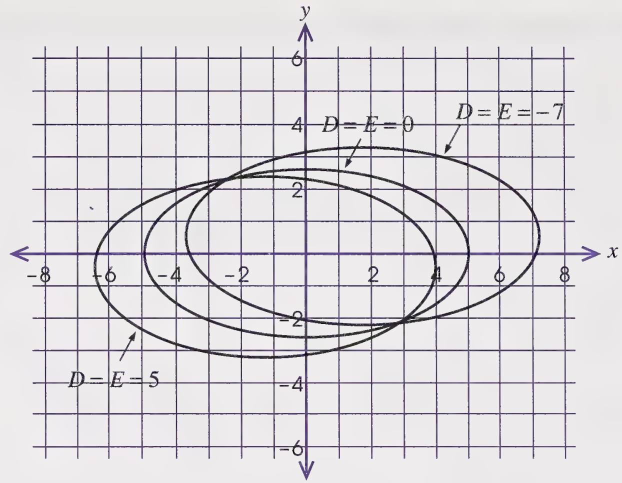
$E = 0$ puts the vertex of the parabola at the origin.

3.



$D = E = 0$ centres the circle at the origin.

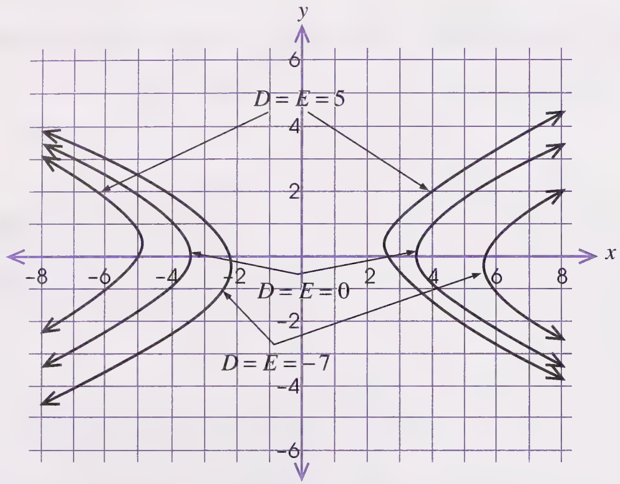
4.



a. $D = E = 0$ centres the ellipse at the origin.

b. The ellipse is centred at the origin when $D = E = 0$.

5.



a. $D = E = 0$ puts the centre of the hyperbola at the origin.

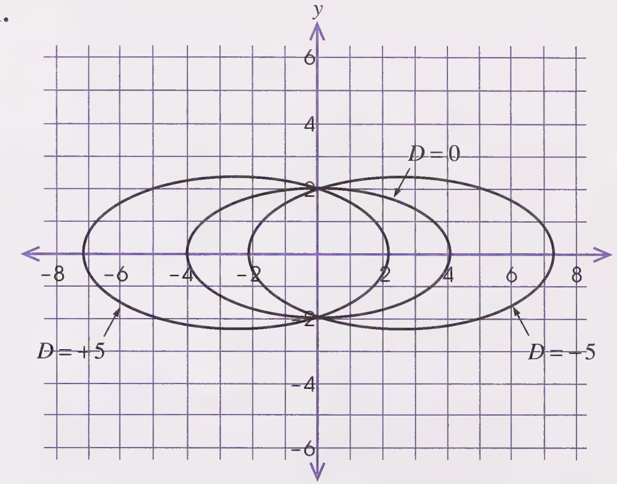
b. $D = E = 0$ centres the hyperbola at the origin.

c. The hyperbola is not centred at the origin if $D = E \neq 0$.

6. When D and E are zero, the centres of a circle, an ellipse, and a hyperbola are located at the origin.

Section 1: Activity 6

1.

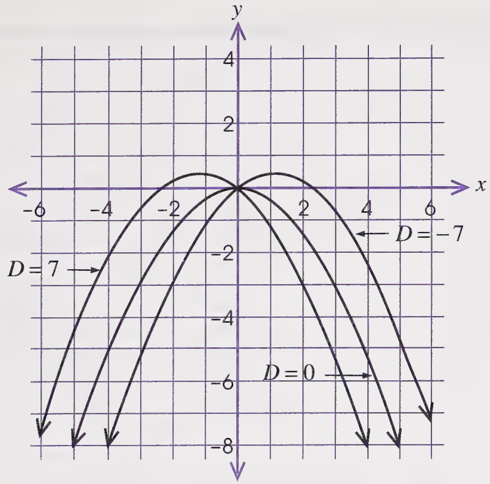


a. The graph moves to the left when the value of D increases.

b. The graph moves to the right when the value of D decreases.

c. Yes, the effect is the same when A and C are interchanged.

2.



a. When D increases, the parabola moves left.

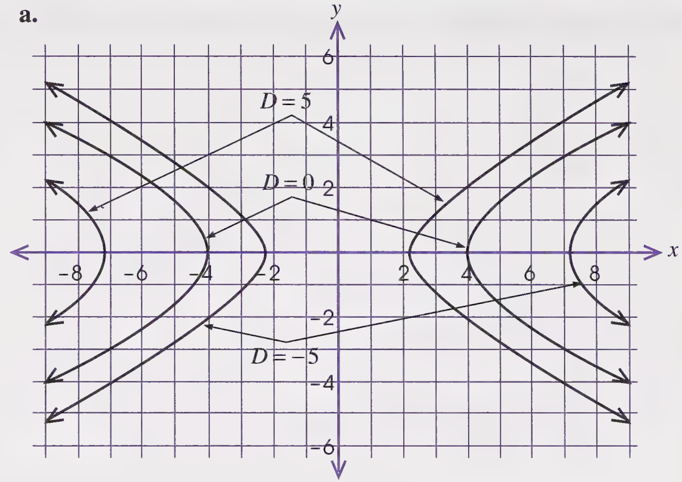
(Did you notice that the graph moved **up** as well as **left**? This is because changing the size of D affects F as well.)

b. When D decreases, the parabola moves right.

(Notice that with a sign change, the movement is only left or right. Hence, for a change in D the dominant effect is for the graph to move left or right.)

c. Yes, the effect is the same for other values of D.

3. a.



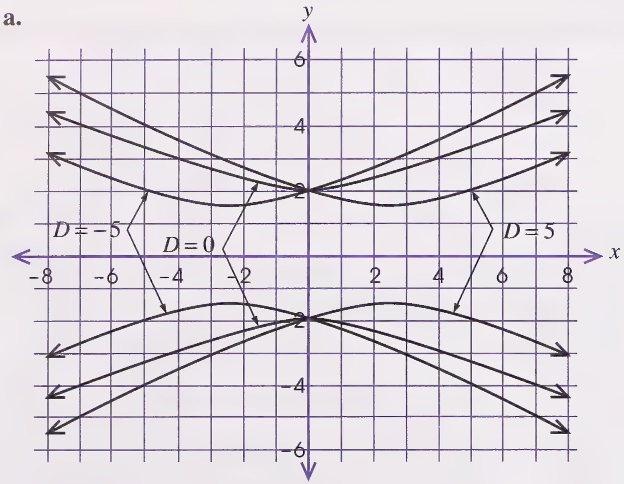
The graph is a hyperbola on the x-axis, centred at the origin.

b. When D increases, the hyperbola moves left.

c. When D decreases, the hyperbola moves right.

d. Yes, the effect is the same for other values of D.

4. a.



This is a hyperbola on the y-axis.

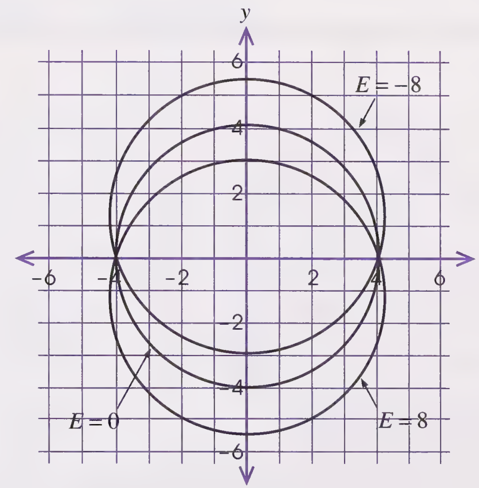
b. When D increases, the hyperbola moves right.

c. When D decreases, the hyperbola moves left.

d. No, the direction the hyperbola moves is opposite in this case. The sign of A and C affects the way D moves the graph.

5. When the value of D changes, a horizontal shift may occur.

6.

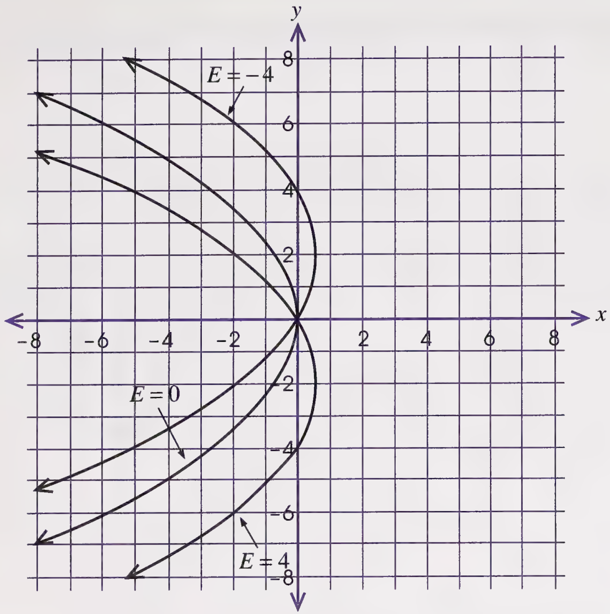


a. When E increases, the circle moves down.

b. When E decreases, the circle moves up.

c. Yes, the effect is the same for other values of E.

7.

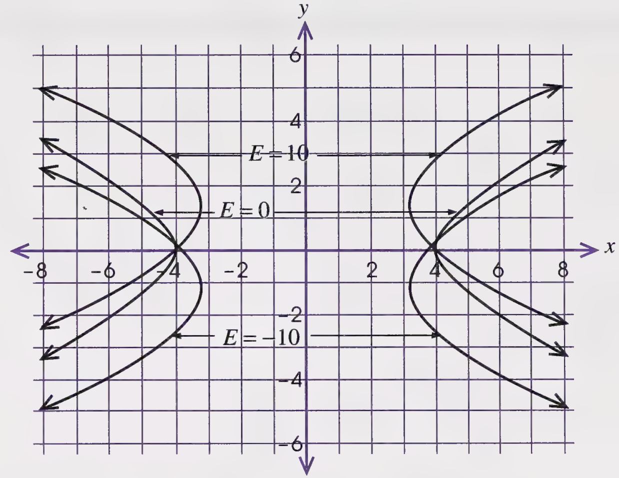


a. The parabola moves down when E increases. The graph moves **right** as well as **up** because changing the size of E affects F, just as changing D affects F.

b. The graph moves up when E decreases. With a sign change, the graph only moves up and down. Hence, for a change in E the dominant effect is up or down.

c. Yes, the effect is the same for other values of E.

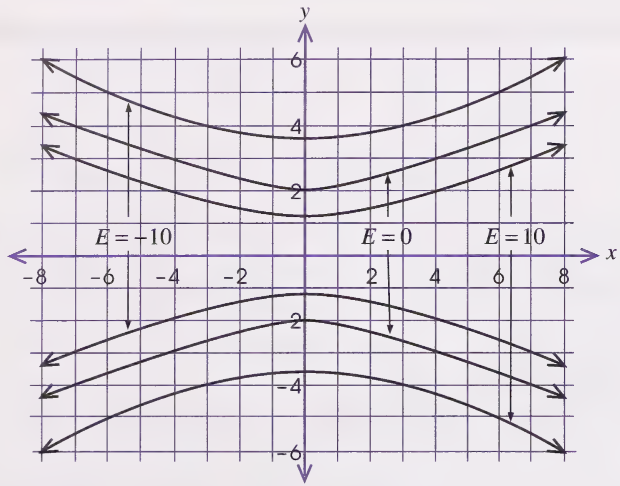
8.



a. When E increases, the graph moves up; when E decreases, the graph moves down.

b. Yes, the effect is the same for other values of E.

9.



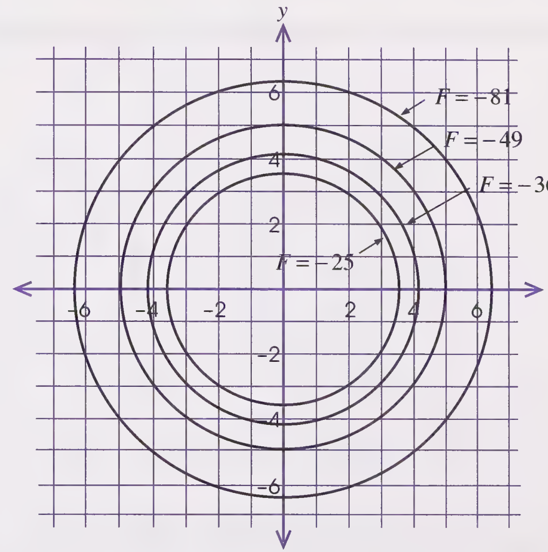
a. The changes expected are as follows:

- The graphs are along the y-axis.
- The graph moves down when E is positive.
- The graph moves up when E is negative.

b. The movement of the graphs is in the opposite direction for each value of E.

10. Changing the value of E moves the graph up or down. Positive E values move the graph down for the circle and ellipse; the graph also moves up for the parabola and hyperbola when A is negative. Negative E values move these graphs up. When A is positive, the effects of E are opposite.

11.



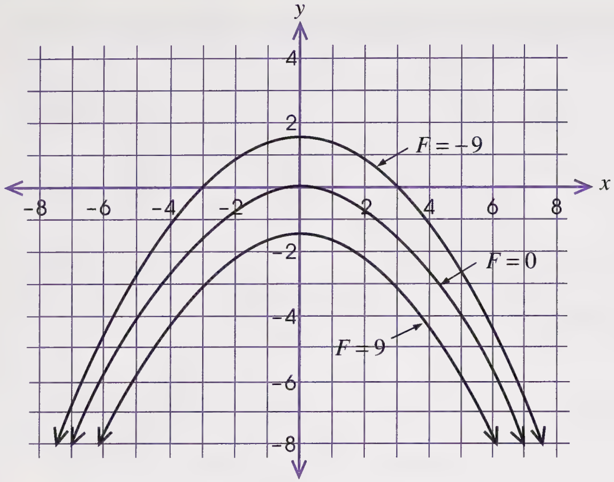
a. As F decreases, the circle becomes larger.

b. Yes, but the circle gets smaller and smaller. If you try negative values less than 0.5, the resolution of the calculator will not allow a graph to be displayed. However, there is still a graph. You could change the range and get a graph which can be seen, but the radius of the circle is still getting smaller.

c. No graph results when $F = 64$.

d. No two real numbers will give you a negative value when they are squared and added together.

12.

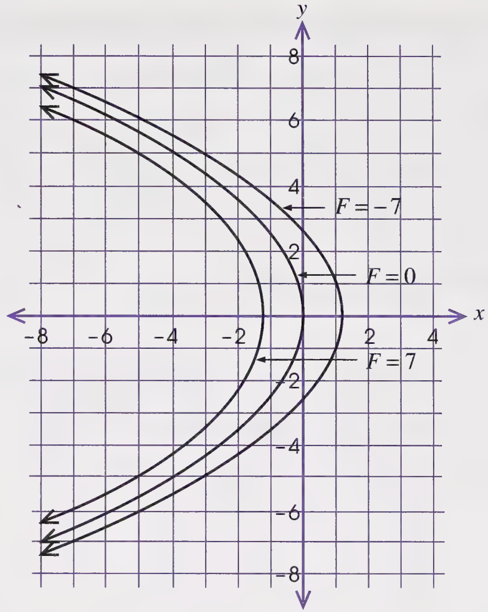


a. The parabola moves down when F increases.

b. The parabola moves up when F decreases.

c. Yes, the effect is the same for other values of F.

13.

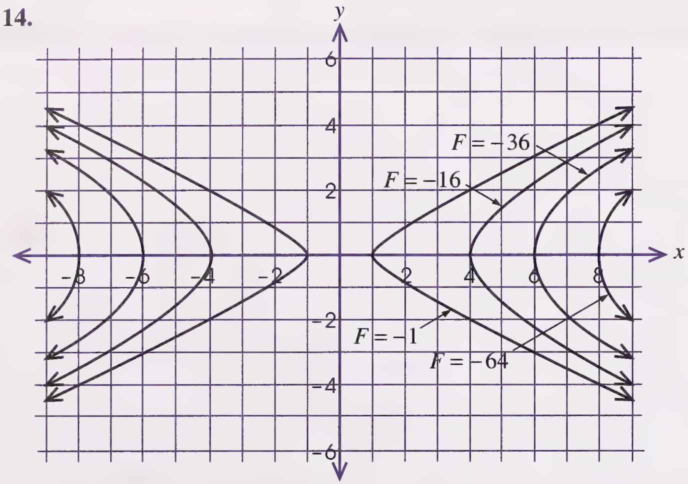


a. When F increases in positive value, the parabola moves left.

b. When F increases in negativity, the parabola moves right.

c. Yes, the effect is the same for other values of F.

14.

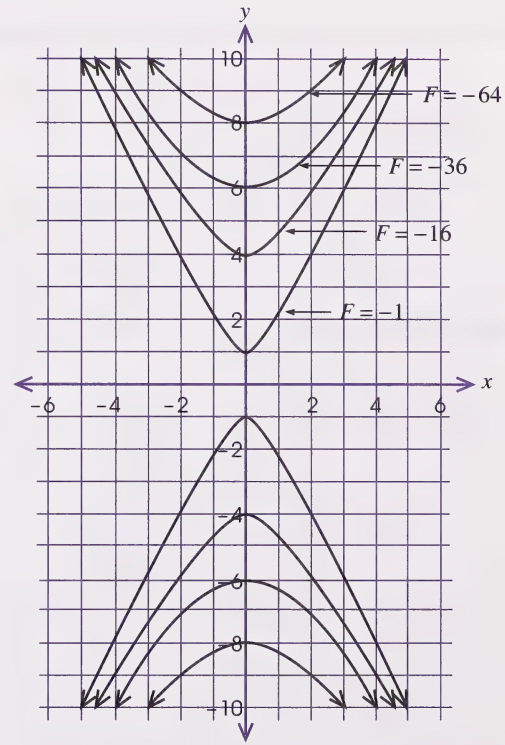


a. As F decreases, the two halves of the hyperbola get further apart along the x-axis and the arms (branches) flatten out.

b. The graph rotates 90° to be on the y-axis. When the equation is multiplied by -1, it becomes the same equation as a hyperbola on the y-axis.

c. As F increases, the two halves get further apart along the y-axis and the arms flatten out.

15.

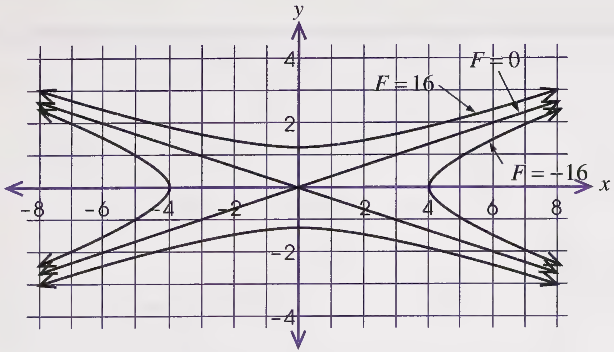


a. The two halves of the hyperbola spread apart and the arms flatten out.

b. The graph rotates 90° to be on the x-axis. If you multiply the equation by -1, it becomes the same equation as a hyperbola on the x-axis.

c. The arms spread apart and flatten out.

16.



a. The graph rotates 90° from the y-axis to the x-axis.

b. The graph becomes two intersecting lines.

$$x^2 - 9y^2 + 0 = 0$$
$$(x-3y)(x+3y) = 0$$
$$x - 3y = 0 \quad \text{or} \quad x + 3y = 0$$
$$y = \frac{1}{3}x \qquad\qquad y = -\frac{1}{3}x$$

The equations of the two intersecting lines are $y = \frac{1}{3}x$ and $y = -\frac{1}{3}x$. The two intersecting lines are another example of a degenerate conic.

c. This is the situation where the plane cuts the two cones on the axis of symmetry.

17. In the case of a circle and an ellipse, the larger the absolute value of F, the larger the curve. In this case of the parabola, F moves the graph up or down when the parabola is on the y-axis. F moves the graph left or right when the parabola is on the x-axis. In the case of the hyperbola, as the absolute value of F increases, the halves of the hyperbola get further apart and the arms flatten. A change from positive values of F to negative values, or vice versa, will switch the axis on which the hyperbola is situated.

Section 1: Follow-up Activities

Extra Help

1. A spotlight forms an ellipse on the ice. The shape is standard.

2. The path of a baseball which is hit is a parabola. This is a standard shape.

3. The path of travel of the comet is hyperbolic.

4. Answers may vary.

a. $A = -3$, $B = 0$, $C = 6$, $D = 0$, $E = 0$, and $F = -16$

A possible equation is $-3x^2 + 6y^2 - 16 = 0$. The reasons which make this a possible equation are as follows:

- Either A or C is negative, and A and C are not equal since the curve is a hyperbola.
- A and F are negative since the curve is on the y-axis.

- If A and F are positive, this type of graph will also be given.
- D and E equal zero since the hyperbola is on the origin.

b. $A = 4$, $B = 0$, $C = 4$, $D = -6$, $E = 8$, and $F = -36$

A possible equation is $4x^2 + 4y^2 - 6x + 8y - 36 = 0$. The reasons which make this a possible equation are as follows:

- $A = C$ since the curve is a circle.
- $B = 0$ since the curve is a circle.
- D is negative and E is positive since the centre is in the fourth quadrant.
- F is negative.
- All parameters could have the opposite sign, but this would really be the same equation.

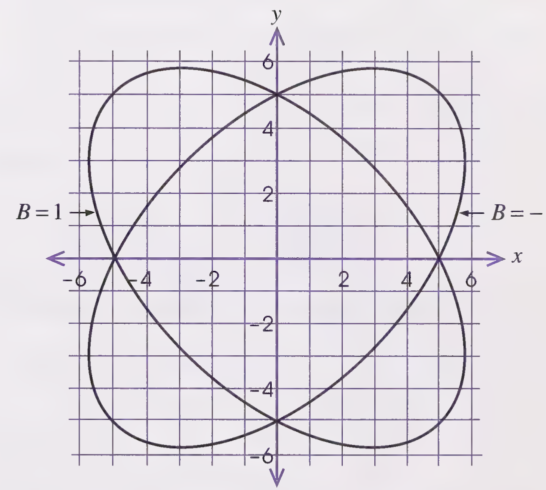
c. $A = 0$, $B = 0$, $C = -7$, $D = -3$, $E = -5$, and $F = 20$

A possible equation is $-7y^2 - 3x - 5y + 20 = 0$. The reasons which make this a possible equation are as follows:

- $A = 0$ since the graph is on the x-axis.
- $B = 0$ since the curve is not rotated.
- D and E are negative since the vertex is in the fourth quadrant. (Remember that D has the opposite effect with a parabola when C is negative.)
- F is most likely positive since the vertex is to the right of the y-axis.

Enrichment

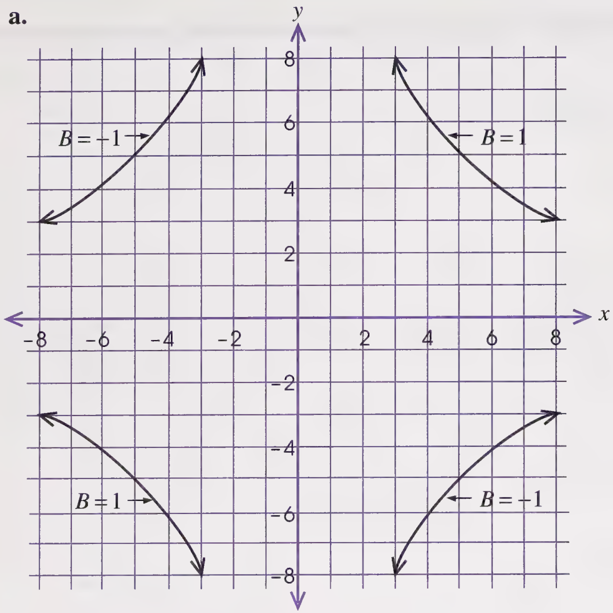
1.



When $B = 1$, the ellipse rotated 135° with the longer axis.

When $B = -1$, the ellipse rotated 45° with the longer axis.

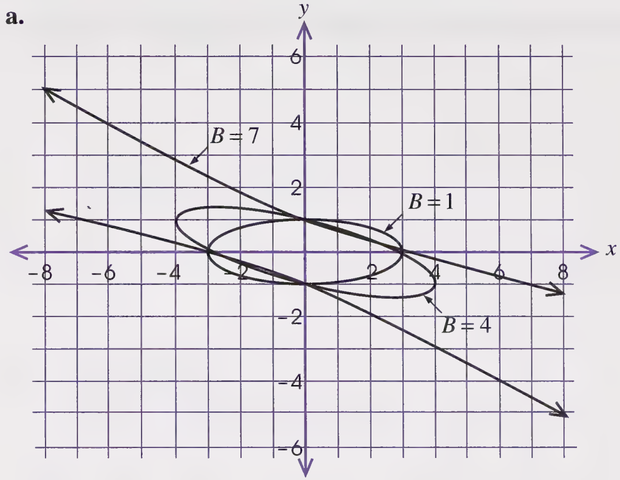
2. a.



The resulting curves are hyperbolas.

b. The position of the hyperbola is rotated 90°.

3. a.



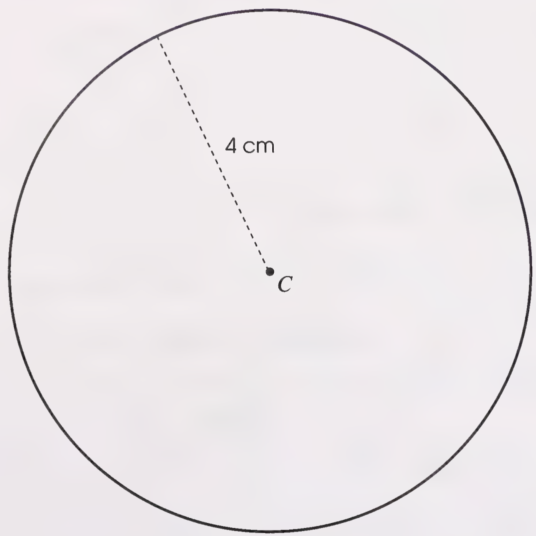
When $B = 1$, you get an ellipse centred on the origin and slightly rotated.

b. When $B = 4$, you get an ellipse centred on the origin and rotated slightly more. It also elongates.

c. When $B = 7$, the curve is a hyperbola.

Section 2: Activity 1

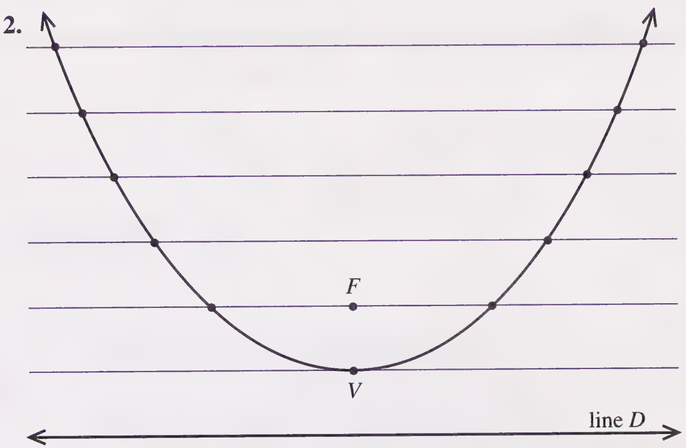
1.



Rule: All of the points are 4 cm from a fixed point C.

Locus definition: A **circle** is the locus of all points in a plane equidistant from a fixed point called the centre of the circle.

2.

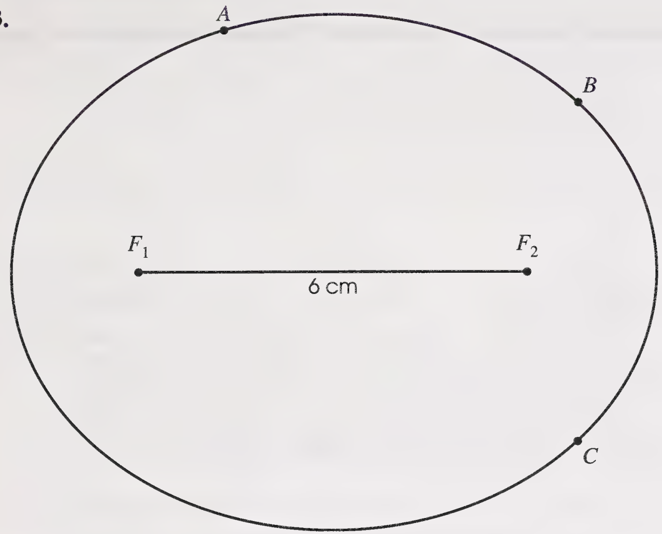


The curve is a parabola where $d(P_1F) = d(P_1D)$. (**Note:** P_1 represents any point on the parabola.)

Rule: Each point on the curve is equidistant from F and D.

Locus definition: A **parabola** is the locus of all points in a plane which are equidistant from a fixed point (focus) and a fixed line (directrix).

3.



a. The curve is an ellipse.

b. $d(AF_1)+d(AF_2)=10 \text{ cm}$

$d(BF_1)+d(BF_2)=10 \text{ cm}$

$d(CF_1)+d(CF_2)=10 \text{ cm}$

c. The sum of the distances from a point on the curve to the fixed points is 10 cm.

d. **Rule:** The sum of the distances from a point on the curve to the fixed points, F_1 and F_2, is a constant.

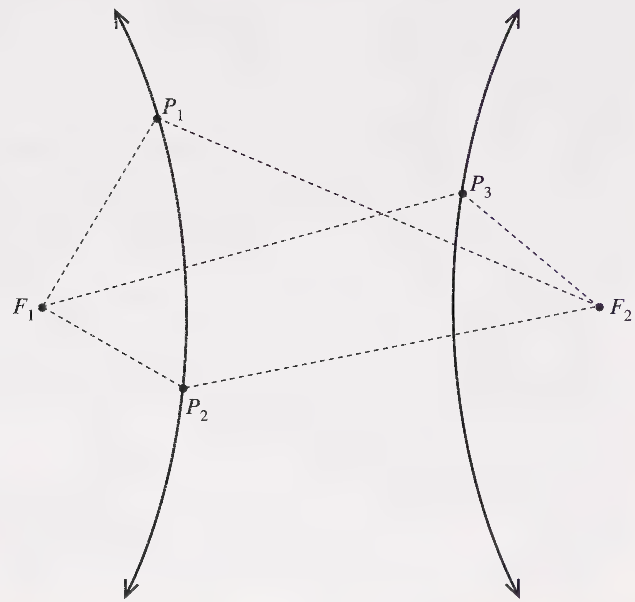
e. **Locus definition:** An **ellipse** is the locus of all points in a plane such that the sum of the distances from any point on the curve to two fixed points, called the **foci**, is constant.

f. If the string is shortened, the ellipse is smaller and more elongated in shape. If the string is longer, the ellipse is larger and assumes a more rounded shape.

g. As the holes get closer, the curve becomes more and more round.

h. If $F_1 = F_2$, the ellipse changes to a circle.

4.



a. The conic is a hyperbola.

b. $d(P_1F_1) - d(P_1F_2) = |3.4 - 7.4|$
$= 4 \text{ cm}$

$d(P_2F_1) - d(P_2F_2) = |2.5 - 6.5|$
$= 4 \text{ cm}$

$d(P_3F_1) - d(P_3F_2) = |6.7 - 2.7|$
$= 4 \text{ cm}$

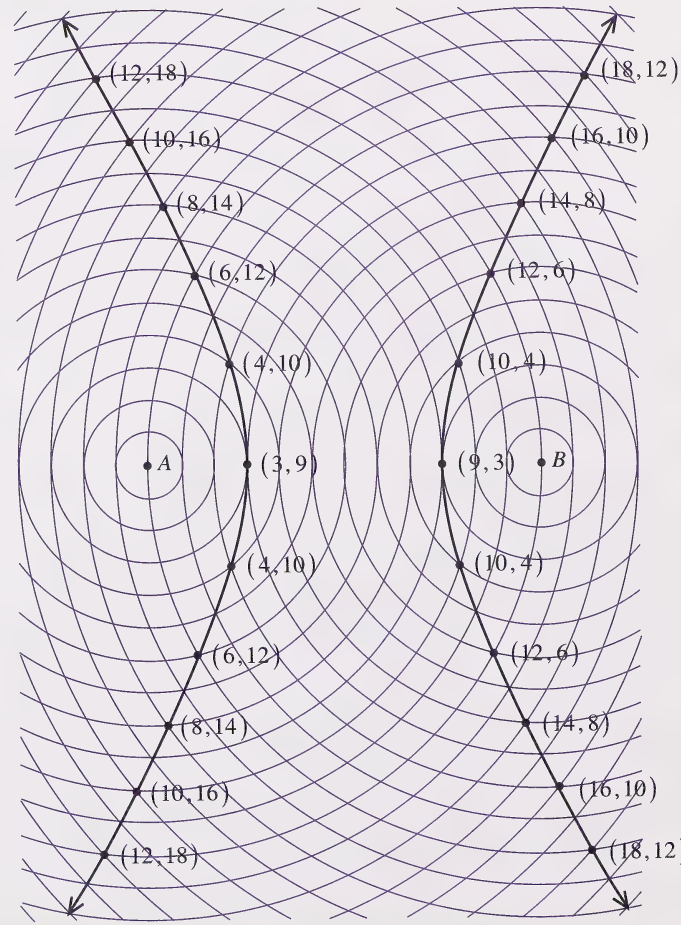
c. The measures are equal.

d. **Rule:** The absolute value of the difference of the distances from a point on the curve to the fixed points, F_1 and F_2, is a constant.

e. **Locus definition:** A **hyperbola** is the locus of all points in a plane such that the absolute value of the difference in distances from any point on the curve to two fixed points, called the **foci**, is constant.

f. The graph is narrower. The distance F_1F_2 must be greater than 4.

5. a. Graphs may vary. The ordered pair (a, b) indicates distances from A and B respectively.



b. The curve is a hyperbola.

c. The curve satisfies the locus definition for a hyperbola. That is, the absolute value of the difference in distances from any point on the curve to two fixed points is constant.

6. a. The value of z is -3.

b. The equation of the directrix is $y=-3$.

c. From the locus definition for a parabola, $d(PF)=d(PD)$. Using the distance formula you can determine its equation.

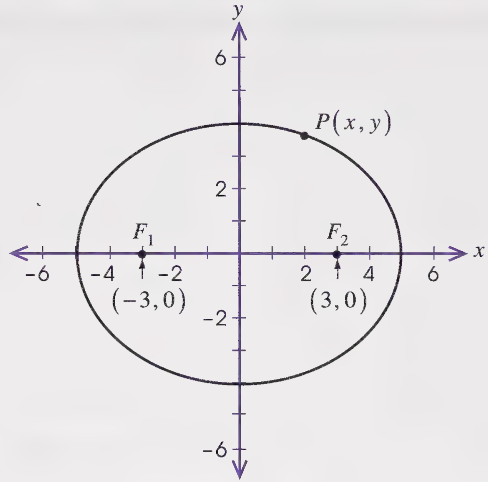
Let $P(x, y)$ be any point on the curve.

The coordinates of F and D are $(0,3)$ and $(x,-3)$.

$$d(PF)=d(PD)$$
$$\sqrt{(x-0)^2+(y-3)^2}=\sqrt{(x-x)^2+\left[y-(-3)^2\right]}$$
$$x^2+(y-3)^2=0+(y+3)^2$$
$$x^2+y^2-6y+9=y^2+6y+9$$
$$x^2-12y=0$$

The equation of the parabola is $x^2-12y=0$ where $A=1$, $B=0$, $C=0$, $D=0$, $E=-12$, and $F=0$.

7.



From the locus definition for an ellipse, $d(PF_1)+d(PF_2)$ is a constant where $F_1(-3,0)$ and $F_2(3,0)$ are foci. Using the distance formula, you can determine its equation.

Let $P(x, y)$ be any point on the curve.

$$d(PF_1)+d(PF_2)=10$$

$$\sqrt{[x-(-3)]^2+(y-0)^2}+\sqrt{(x-3)^2+(y-0)^2}=10$$

$$\sqrt{(x+3)^2+y^2}+\sqrt{(x-3)^2+y^2}=10$$

$$\sqrt{(x+3)+y^2}=10-\sqrt{(x-3)^2+y^2}$$

$$\left(\sqrt{(x+3)^2+y^2}\right)^2=\left(10-\sqrt{(x-3)^2+y^2}\right)^2$$

$$(x+3)^2+y^2=100-20\sqrt{(x-3)^2+y^2}+(x-3)^2+y^2$$

$$x^2+6x+9+y^2=100-20\sqrt{(x-3)^2+y^2}+x^2-6x+9+y^2$$

$$12x-100=-20\sqrt{(x-3)^2+y^2}$$

$$(12x-100)^2=\left(-20\sqrt{(x-3)^2+y^2}\right)^2$$

$$144x^2-2400x+10\,000=400\left(x^2-6x+9+y^2\right)$$

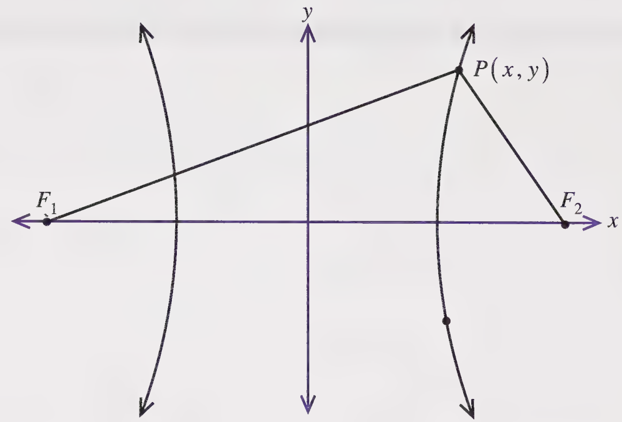
$$144x^2-2400x+10\,000=400x^2-2400x+3600+400y^2$$

$$-256x^2-400y^2+6400=0 \qquad \text{(Divide each term by } -16.)$$

$$16x^2+25y^2-400=0$$

The equation of the ellipse is $16x^2+25y^2-400=0$.

8. From the locus definition for a hyperbola, $\left|d(PF_1)-d(PF_2)\right|$ is a constant. Using the distance formula you can determine its equation.



$P(x, y)$ is any point on the curve. The coordinates of F_1 and F_2 are $(-4, 0)$ and $(4, 0)$.

$$\left|d(PF_1)-d(PF_2)\right|=4$$

$$\sqrt{[x-(-4)]^2+(y-0)^2}-\sqrt{(x-4)^2+(y-0)^2}=\pm 4$$

$$\sqrt{(x+4)^2+y^2}=\sqrt{(x-4)^2+y^2}\pm 4$$

$$\left(\sqrt{(x+4)^2+y^2}\right)^2=\left(\sqrt{(x-4)^2+y^2}\pm 4\right)^2$$

$$(x+4)^2+y^2=(x-4)^2+y^2\pm 8\sqrt{(x-4)^2+y^2}+16$$

$$x^2+8x+16+y^2=x^2-8x+16+y^2\pm 8\sqrt{x^2-8x+16+y^2}+16$$

$$16x-16=\pm 8\sqrt{x^2-8x+16+y^2}$$

$$2x-2=\pm\sqrt{x^2-8x+16+y^2}$$

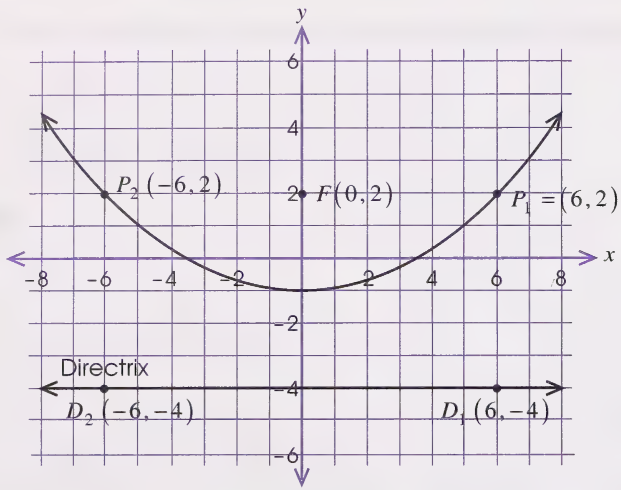
$$(2x-2)^2=\left(\pm\sqrt{x^2-8x+16+y^2}\right)^2$$

$$4x^2-8x+4=x^2-8x+16+y^2$$

$$3x^2-y^2-12=0$$

The equation of the hyperbola is $3x^2-y^2-12=0$.

9.



Substitute a suitable value for y in the equation $x^2 - 12y - 12 = 0$, and solve for x.

Let $y = 2$.

$$x^2 - 12(2) - 12 = 0$$
$$x^2 = 36$$
$$x = \pm 6$$

The two points on the curve are $(-6, 2)$ and $(6, 2)$. By the locus definition for a parabola, $d(PF) = d(PD)$. Using the distance formula and the points $P_1(6, 2)$, $F(0, 2)$, and $D_1(6, -4)$, you can verify the locus definition for this parabola.

$$d(P_1F) = d(P_1D_1)$$
$$\sqrt{(6-0)^2 + (2-2)^2} = \sqrt{(6-6)^2 + [2-(-4)]^2}$$
$$\sqrt{(6)^2 + 0^2} = \sqrt{0^2 + (6)^2}$$
$$\sqrt{36} = \sqrt{36}$$
$$6 = 6$$

Now use the points $P_2(-6, 2)$, $F(0, 2)$, and $D_2(-6, -4)$ in the distance formula.

$$d(P_2F) = d(P_2D_2)$$
$$\sqrt{(-6-0)^2 + (2-2)^2} = \sqrt{[-6-(-6)]^2 + [2-(-4)]^2}$$
$$\sqrt{(-6)^2 + 0^2} = \sqrt{0^2 + (6)^2}$$
$$\sqrt{36} = \sqrt{36}$$
$$6 = 6$$

In both cases $d(PF) = d(PD)$.

10. a. The locus for a seat on a revolving Ferris wheel is a circle.

b. The locus for a spider at the end of a swinging pendulum is an arc.

c. The locus for the hub of a wheel on a flat surface is a straight line.

11. $d(PN)-d(PM)=100-50$

$$=50$$

$$\sqrt{(x-x)^2+(y-0)^2}-\sqrt{(x+x)^2+(y-0)^2}=50$$

$$y-\sqrt{(2x)^2+y^2}=50$$

$$y-50=\sqrt{(2x)^2+y^2}$$

$$(y-50)^2=\left(\sqrt{4x^2+y^2}\right)^2$$

$$y^2-100y+2500=4x^2+y^2$$

$$4x^2+100y-2500=0$$

When $y=0$, $4x^2-2500=0$

$$4x^2=2500$$

$$x^2=625$$

$$x=\pm 25$$

$d(MN)=2\times 25$

$=50$

The two stations are 50 km apart.

12. The locus is a hyperbola.

$$d(PF_1)-d(PF_2)$$

$$=\sqrt{(x-x_1)^2+(y-y_1)^2}-\sqrt{(x-x_2)^2+(y-y_2)^2}$$

Let $(x,y)=(9,6)$.

$$d(PF_1)-d(PF_2)$$

$$=\sqrt{[9-(-4)]^2+(6-0)^2}-\sqrt{(9-4)^2+(6-0)^2}$$

$$=\sqrt{(13)^2+(6)^2}-\sqrt{(5)^2+(6)^2}$$

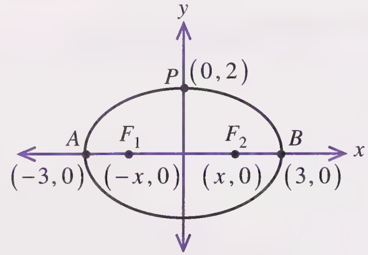
$$=\sqrt{169+36}-\sqrt{25+36}$$

$$=\sqrt{205}-\sqrt{61}$$

$$=6.5$$

The constant of difference from $(9,6)$ to the fixed points F_1 and F_2 is not equal to 6. Therefore, $(9,6)$ does not lie on the curve.

13.



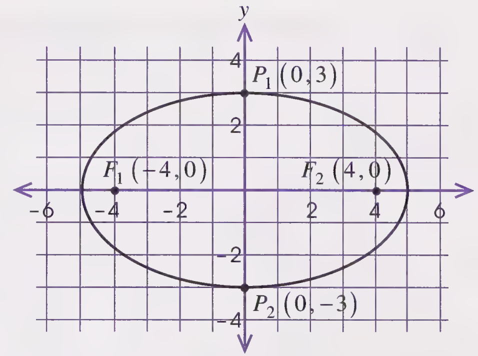
a. The cord should be 6 m long.

b. $d(BF_1) + d(BF_2) = d(AB)$
$= 6$

$$\begin{aligned} d(PF_1) + d(PF_2) &= 6 \\ \sqrt{(0+x)^2 + (2-0)^2} + \sqrt{(0-x)^2 + (2-0)^2} &= 6 \\ \sqrt{x^2+4} + \sqrt{x^2+4} &= 6 \\ 2\sqrt{x^2+4} &= 6 \\ \sqrt{x^2+4} &= 3 \\ \left(\sqrt{x^2+4}\right)^2 &= 3^2 \\ x^2+4 &= 9 \\ x^2 &= 5 \\ x &= \pm\sqrt{5} \end{aligned}$$

The peg should be $\sqrt{5}$ m from the centre.

14.



Let $x = 0$.

$$\begin{aligned} \therefore\ 25y^2 - 225 &= 0 \\ 25y^2 &= 225 \\ y^2 &= 9 \\ y &= \pm 3 \end{aligned}$$

The two points are $P_1\,(0, 3)$ and $P_2\,(0, -3)$.

Let $k = d(P_1F_1) + d(P_1F_2)$

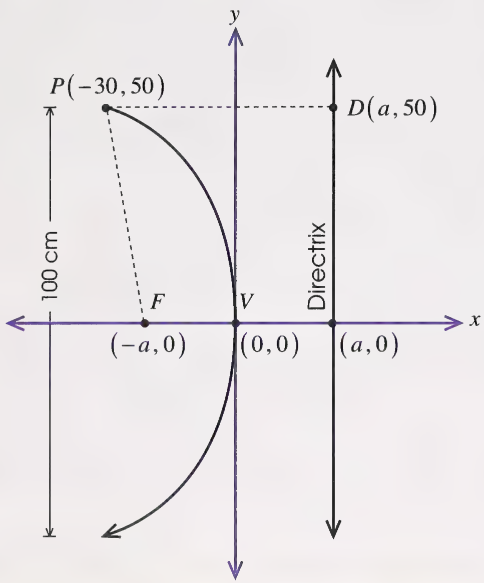
$$\begin{aligned} k &= \sqrt{[0-(-4)]^2 + (3-0)^2} + \sqrt{(0-4)^2 + (3-0)^2} \\ &= \sqrt{(4)^2 + (3)^2} + \sqrt{(-4)^2 + (3)^2} \\ &= \sqrt{16+9} + \sqrt{16+9} \\ &= 5+5 \\ &= 10 \end{aligned}$$

Let $\ell = d(P_2F_1) + d(P_2F_2)$

$$\ell = \sqrt{[0-(-4)]^2 + (-3-0)^2} + \sqrt{(0-4)^2 + (-3-0)^2}$$
$$= \sqrt{(4)^2 + (-3)^2} + \sqrt{(-4)^2 + (-3)^2}$$
$$= \sqrt{16+9} + \sqrt{16+9}$$
$$= 5+5$$
$$= 10$$

Since $k = \ell$, you can state for this ellipse that $d(PF_1) + d(PF_2) = 10$.

15.

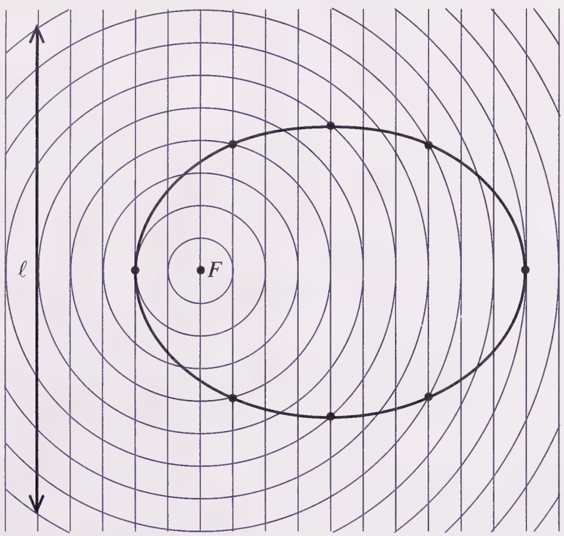


$$d(PF) = d(PD)$$
$$\sqrt{[-30-(-a)]^2 + (50-0)^2} = \sqrt{(-30-a)^2 + (50-50)^2}$$
$$(-30+a)^2 + (50)^2 = (-30-a)^2 + 0^2$$
$$900 - 60a + a^2 + 2500 = 900 + 60a + a^2$$
$$120a = 2500$$
$$a = \frac{2500}{120}$$
$$= 20.8$$

The distance from the vertex to the light source is 20.8 cm.

Section 2: Activity 2

1. The graph will vary. The following graph is just one example. The conic is an ellipse.



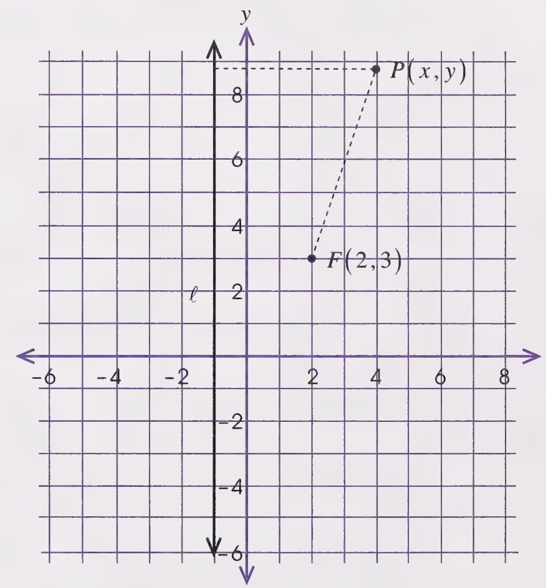
2. Graphs will vary. The larger eccentricity produces bigger and more elongated ellipses. When e is close to zero, the ellipse is close to a circle. If $0 < e < 1$, then the conic is an ellipse.

3. Graphs will vary. The ellipses are similar in shape. If the distance between the directrix and focus is larger, then the ellipse will be larger in size.

4. As the eccentricity gets larger, the arms of the hyperbola become wider.

5. If you move the directrix to the right, the hyperbola will be moved to the right. If you move the directrix to the left, the hyperbola will be moved to the left. Although the focus and eccentricity would remain the same, the hyperbola would not be the same.

6.



$$e = \frac{\text{distance from } P \text{ to } F}{\text{distance from } P \text{ to } l}$$

$$\frac{3}{2} = \frac{\sqrt{(x-2)^2 + (y-3)^2}}{x+1}$$

$$3(x+1) = 2\sqrt{(x-2)^2 + (y-3)^2}$$

$$9(x+1)^2 = 4\left[(x-2)^2 + (y-3)^2\right]$$

$$9x^2 + 18x + 9 = 4\left(x^2 - 4x + y^2 - 6y + 13\right)$$

$$9x^2 + 18x + 9 = 4x^2 - 16x + 4y^2 - 24y + 52$$

$$5x^2 - 4y^2 + 34x + 24y - 43 = 0$$

This conic is a hyperbola.

7.

$$1 = \frac{\sqrt{(x-2)^2 + (y-3)^2}}{x+1}$$

$$(x+1)^2 = (x-2)^2 + (y-3)^2$$

$$x^2 + 2x + 1 = x^2 - 4x + 4 + y^2 - 6y + 9$$

$$y^2 - 6x - 6y + 12 = 0$$

This is a parabola.

$$\frac{3}{4} = \frac{\sqrt{(x-2)^2 + (y-3)^2}}{x+1}$$

$$9(x+1)^2 = 16\left[(x-2)^2 + (y-3)^2\right]$$

$$9x^2 + 18x + 9 = 16x^2 - 64x + 64 + 16y^2 - 96y + 144$$

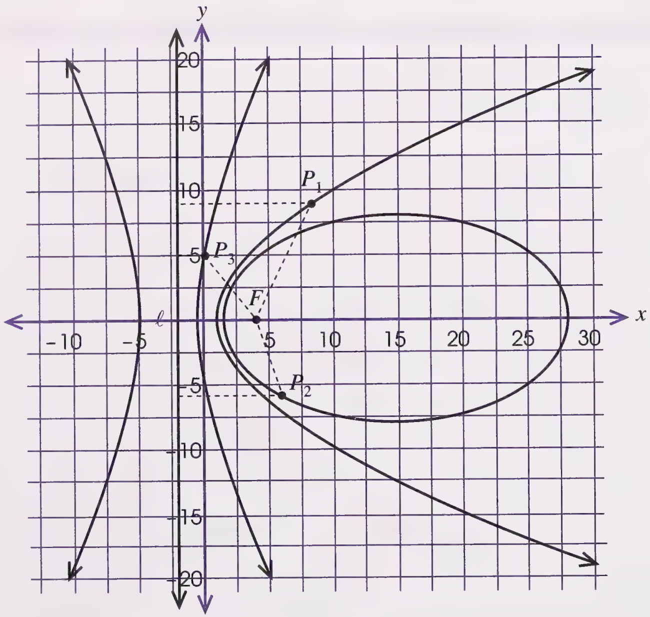
$$7x^2 + 16y^2 - 82x - 96y + 199 = 0$$

This is an ellipse.

Definitions

- If $e = 1$, the conic is a parabola.
- If $0 < e < 1$, the conic is a ellipse.
- If $e > 1$, the conic is a hyperbola.

8. The following diagram shows one possible way to get the answer. You may want to try a different point. The answer should be the same.



The distance from P_1 to F is 2.0 cm.

The distance from P_1 to line ℓ is 2.0 cm.

$$\therefore \frac{d(P_1 F)}{d(P_1 \ell)} = \frac{2.0}{2.0} = 1$$

The eccentricity of the parabola is 1.

The distance from P_2 to F is 1.2 cm.

The distance from P_2 to line ℓ is 1.6 cm.

$$\therefore \frac{d(P_2 F)}{d(P_2 \ell)} = \frac{1.2}{1.6} \doteq 0.8$$

The eccentricity of the ellipse is approximately 0.8.

The distance from P_3 to F is 1.3 cm.

The distance from P_3 to line ℓ is 0.4 cm.

$$\therefore \frac{d(P_3 F)}{d(P_3 \ell)} = \frac{1.3}{0.4} \doteq 3.3$$

The eccentricity of the hyperbola is approximately 3.3.

9. The eccentricity does not affect the shape of a parabola. If the focus is closer to the directrix, the parabola would be narrower and its vertex would be closer to the directrix.

10. If the directrix and focus remain the same, then a larger ellipse would have a larger eccentricity.

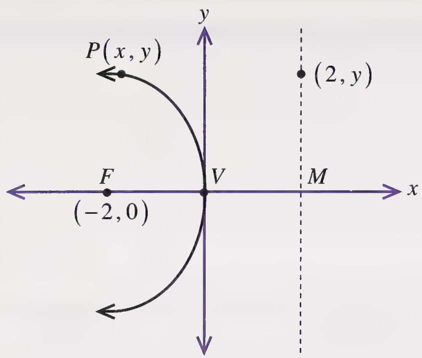
11. If the directrix and focus remain the same, the hyperbola with a smaller eccentricity would be narrower and the two sections of the hyperbola would be farther apart.

12. The eccentricity of the parabola is 1.

Since $e = 1$, $d(FV) = d(VM)$.

Therefore, the coordinates of M are $(2, 0)$.

The directrix is $x = 2$.



$$\frac{d(PF)}{\text{distance from } P \text{ to the directrix}} = 1$$

$$\frac{\sqrt{(x+2)^2 + (y-0)^2}}{x-2} = 1$$

$$\sqrt{(x+2)^2 + y^2} = x - 2$$

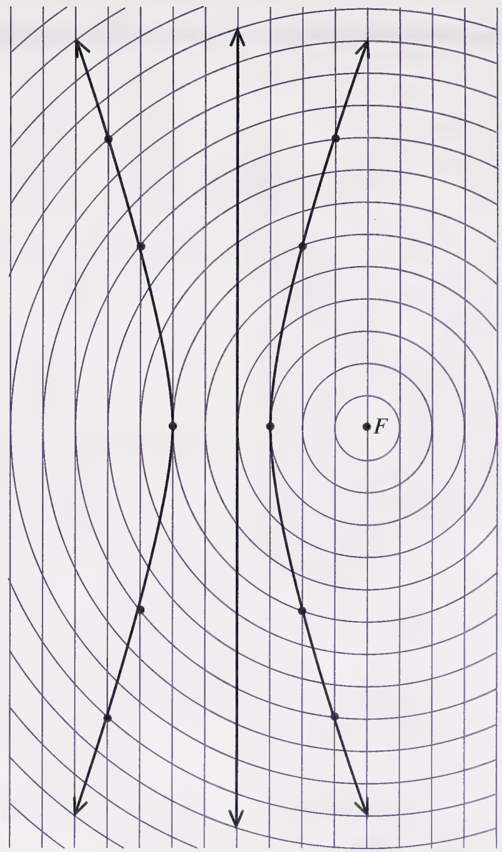
$$(x+2)^2 + y^2 = (x-2)^2$$

$$x^2 + 4x + 4 + y^2 = x^2 - 4x + 4$$

$$y^2 + 8x = 0$$

The equation of the reflecting surface is $y^2 + 8x = 0$.

13. a.



b. The conic is a hyperbola.

14. $$\frac{\text{distance between } P \text{ and the focus}}{\text{distance between } P \text{ and the directrix}} = \frac{\sqrt{(4-5)^2+(0-0)^2}}{4-\frac{16}{5}}$$
$$= \frac{1}{\left(\frac{4}{5}\right)}$$
$$= \frac{5}{4}$$

Since the eccentricity of the conic is $\frac{5}{4}$, point $(4, 0)$ is on the conic.

Section 2: Follow-up Activities

Extra Help

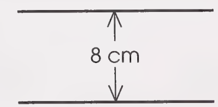
1. a. The locus of taking an elevator is a vertical line.

   b. The locus of a speck of dust on a revolving record is a circle.

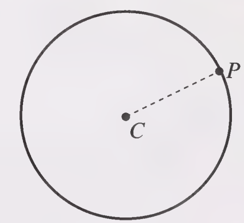
   c. The locus of sitting in the seat of a rotating Ferris wheel is a circle.

   d. The locus of two children playing on a seesaw is two arcs.

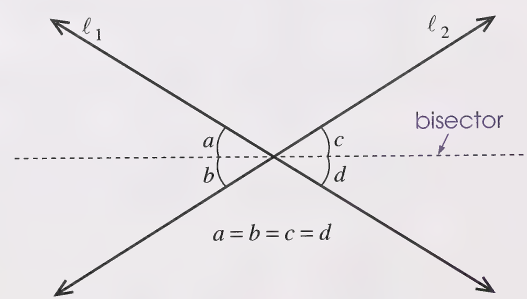
2. a. The locus is a line parallel to the given line and 8 cm from it.



   b. The locus of the path of a ball tied to the end of a rope and swung by a person is a circle with radius PC.



   c. The locus of all points equidistant from two intersecting lines is a line which is a bisector of the angle formed by the two intersecting lines.



Enrichment

1. a. $$\begin{aligned} B^2 - 4AC &= (0)^2 - 4(1)(1) \\ &= -4 \end{aligned}$$

   Since $B^2 - 4AC < 0$, the equation may represent an ellipse, a circle, a point, or no curve.

   b. $$\begin{aligned} B^2 - 4AC &= (0)^2 - 4(0)(5) \\ &= 0 \end{aligned}$$

   Since $B^2 - 4AC = 0$, the curve may be a parabola, two parallel lines, one line, or no curve.

   c. $$\begin{aligned} B^2 - 4AC &= (0)^2 - 4(3)(5) \\ &= -60 \end{aligned}$$

   Since $B^2 - 4AC < 0$, where $A \neq C$ and A and C are both positive, the equation represents an ellipse.

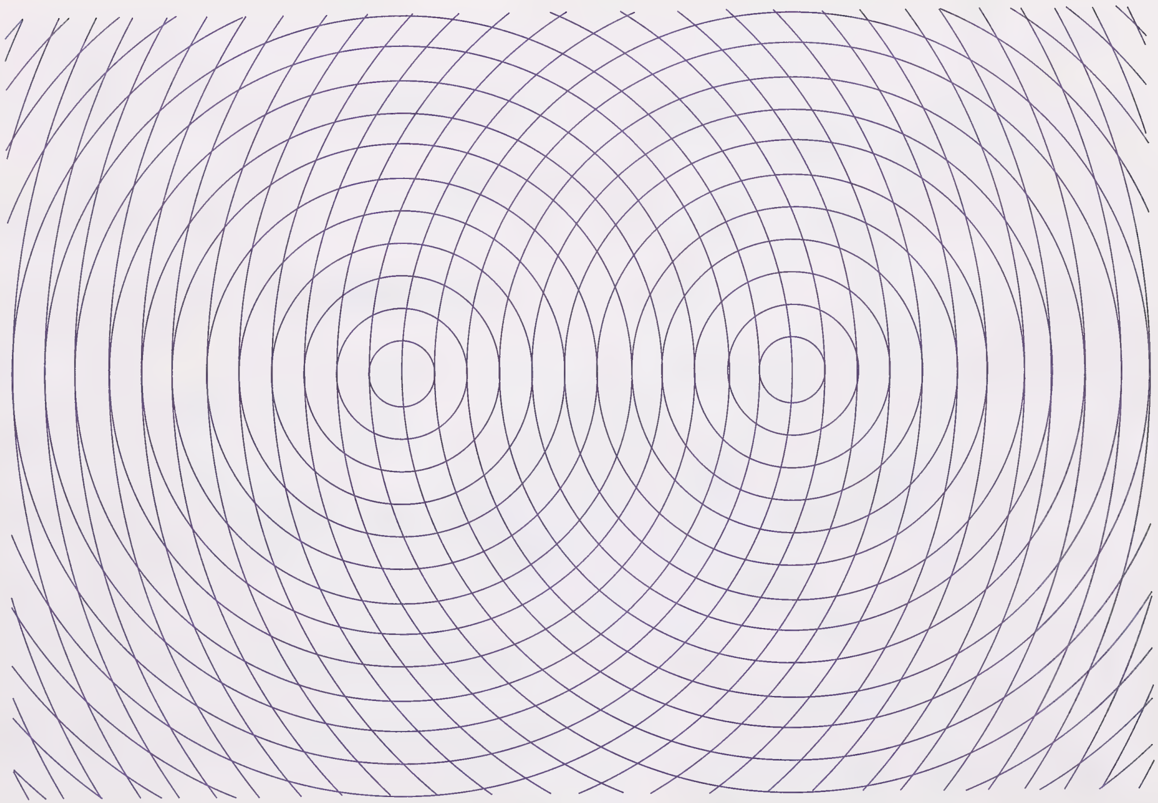
2. a. $$\begin{aligned} B^2 - 4AC &= (0)^2 - 4(2)(2) \\ &= -16 \end{aligned}$$

   Since $B^2 - 4AC < 0$, and $A = C$, the curve is a circle.

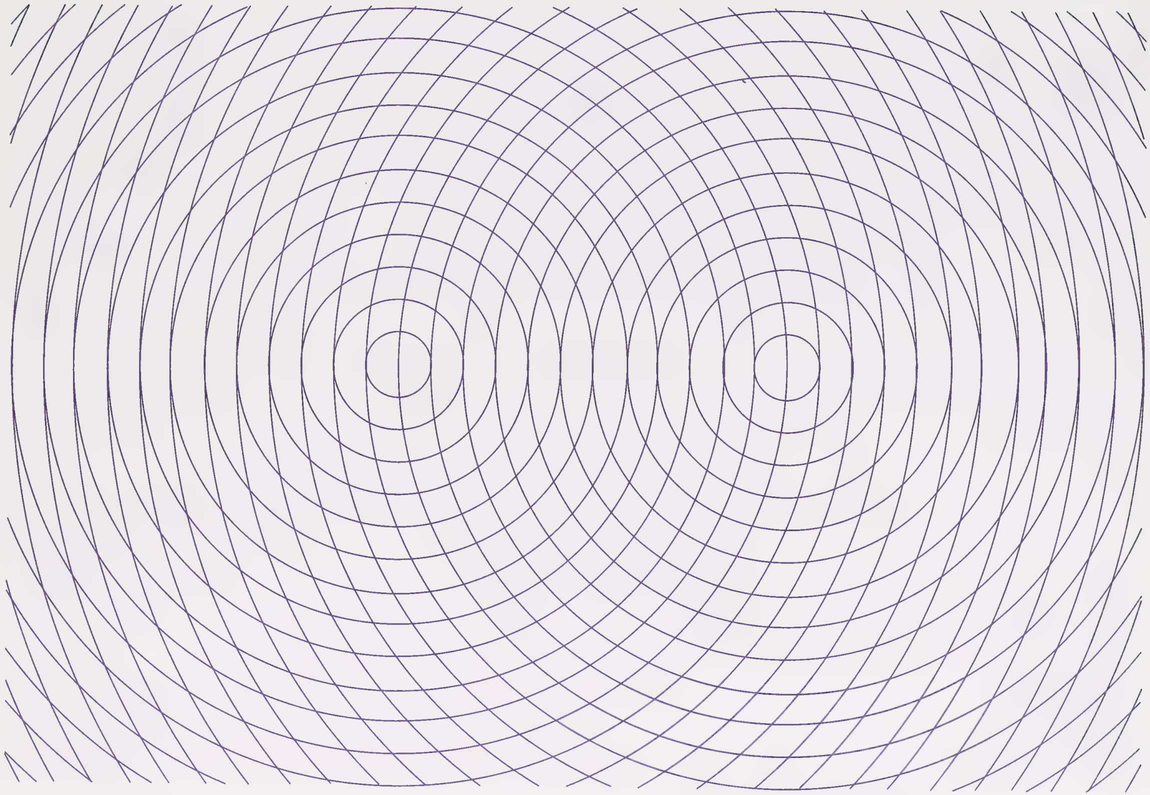
   b. $$\begin{aligned} B^2 - 4AC &= (0)^2 - 4(4)(-9) \\ &= 144 \end{aligned}$$

   Since $B^2 - 4AC > 0$, and A and C have opposite signs, the conic is a hyperbola.

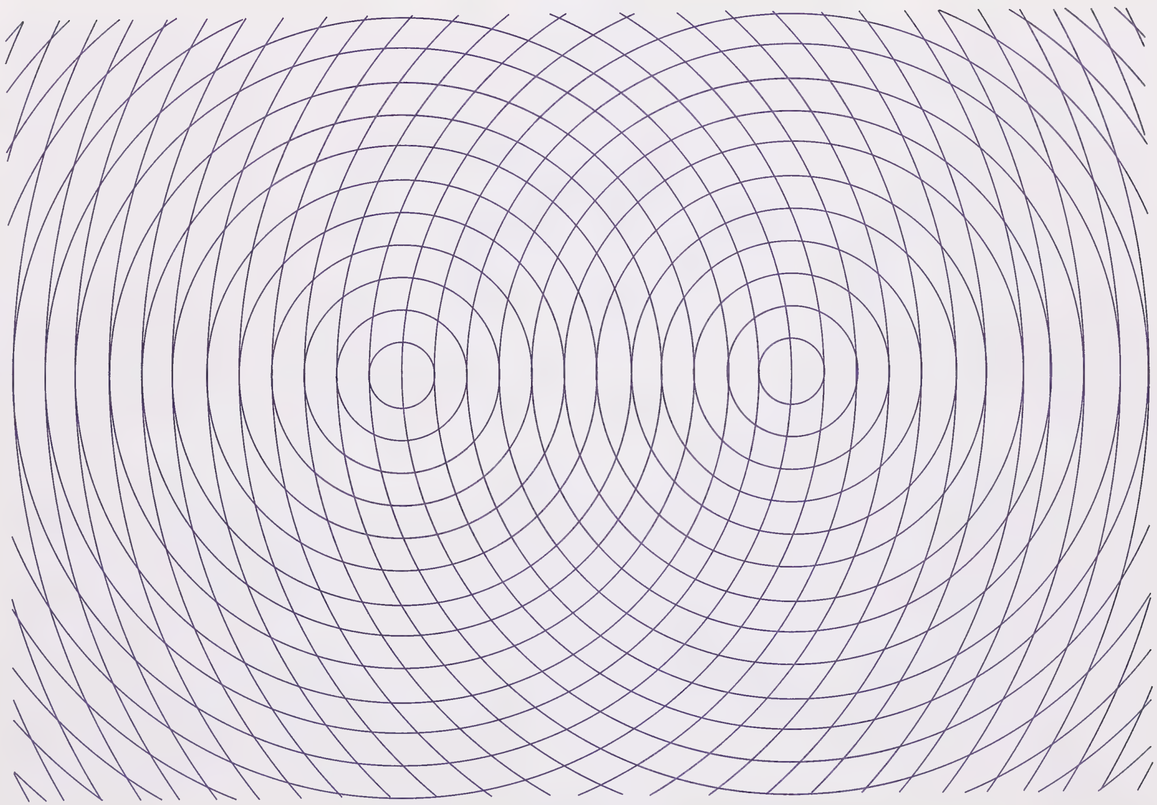
Double-Concentric-Circle Graph Paper



Double-Concentric-Circle Graph Paper



Double-Concentric-Circle Graph Paper



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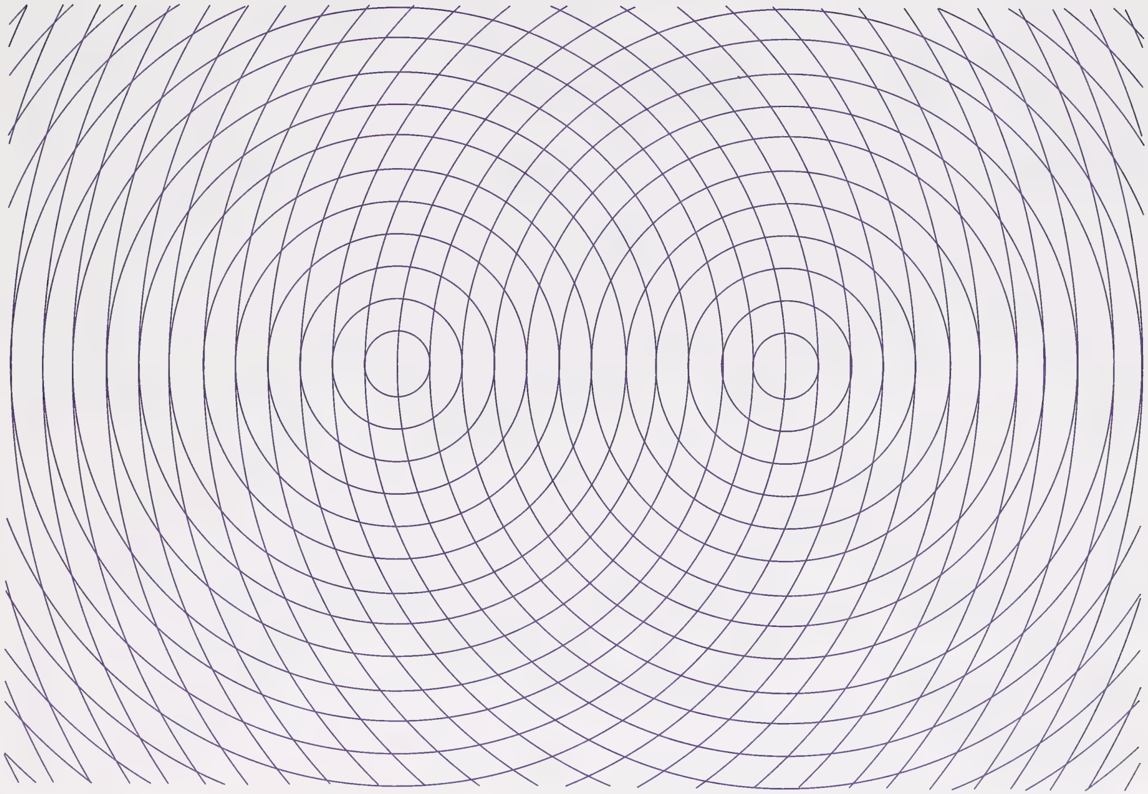
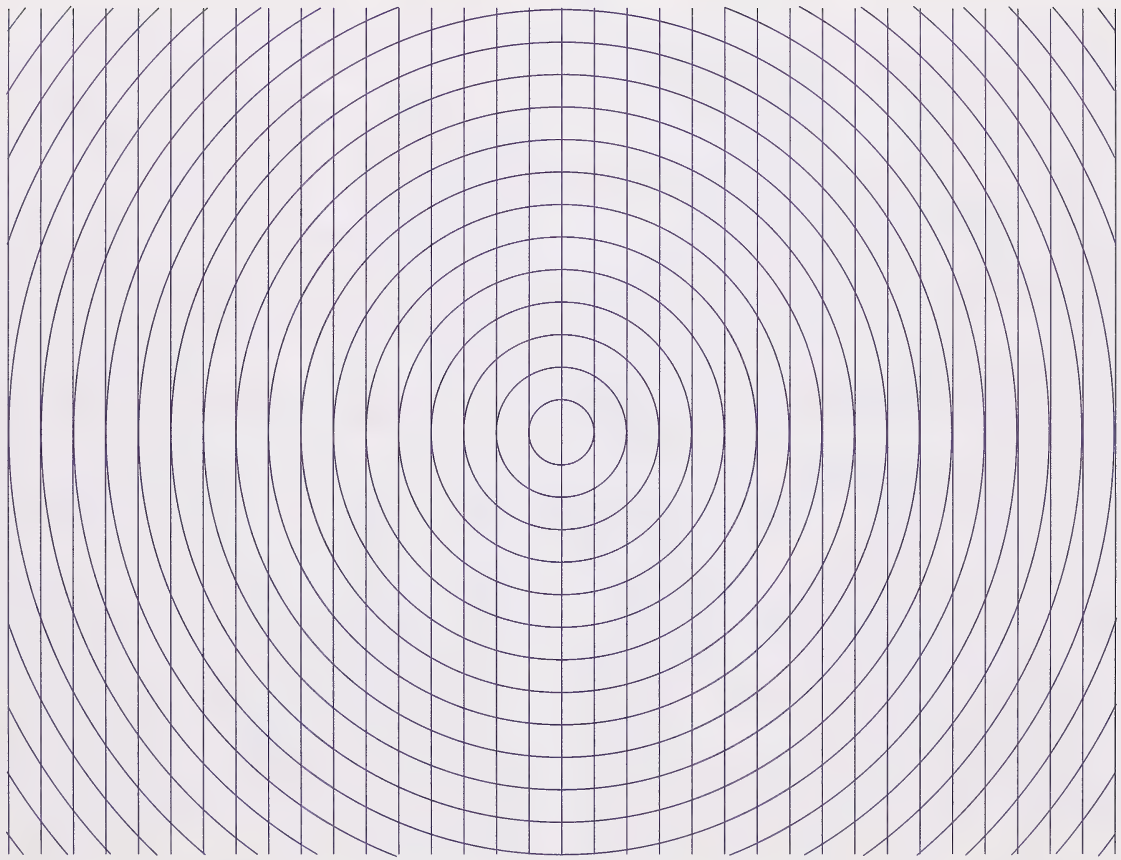
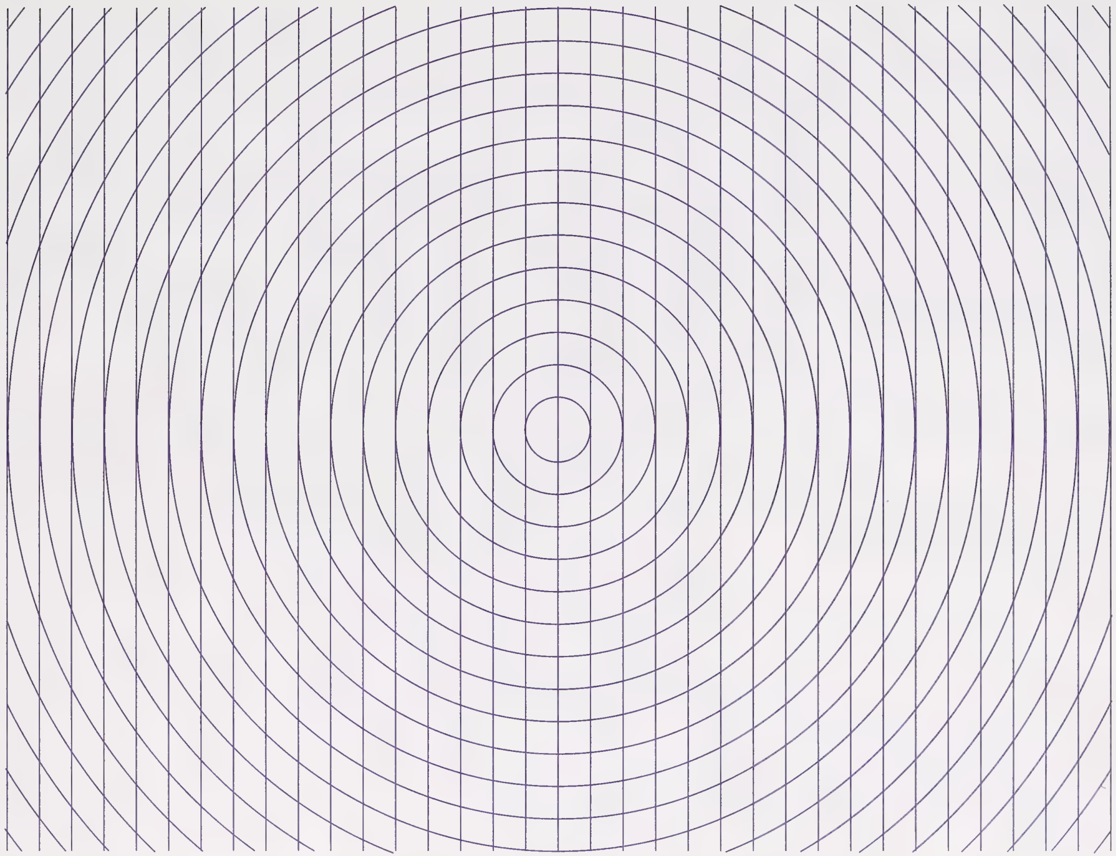


Circle-Line Graph Paper

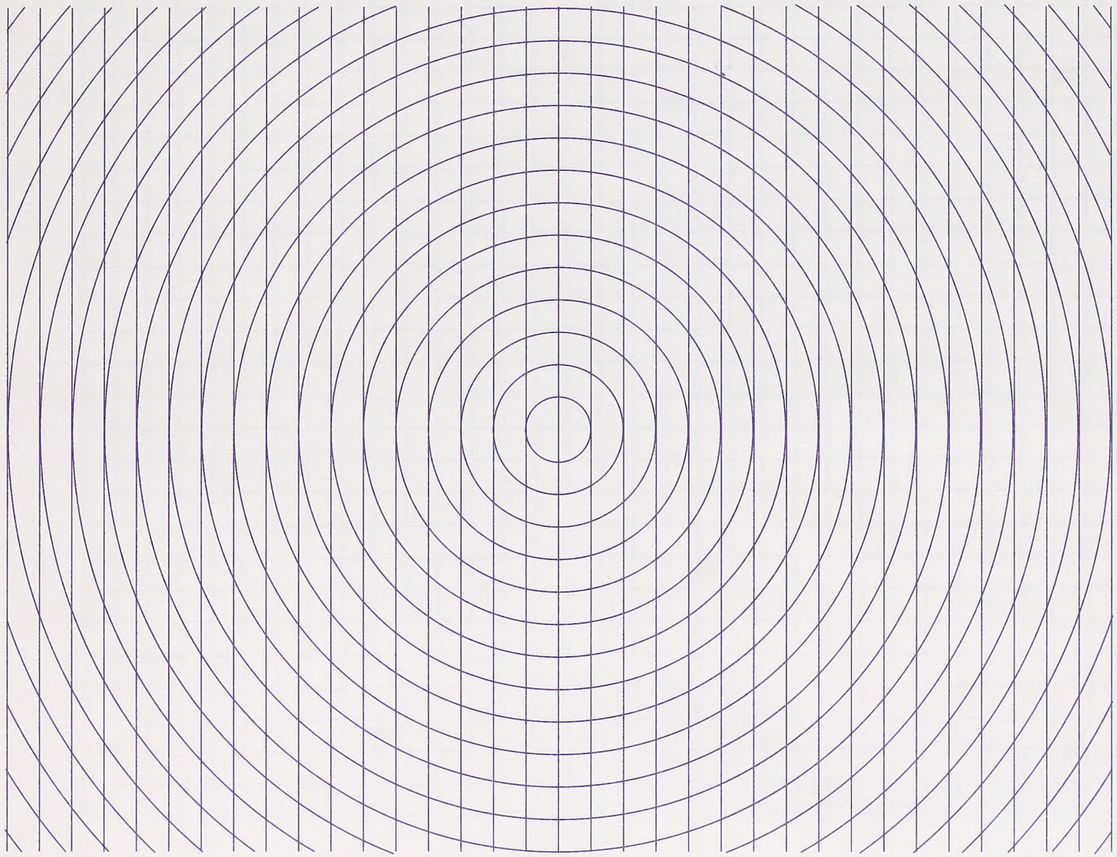


Circle-Line Graph Paper



Circle-Line Graph Paper

Circle-Line Graph Paper



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